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COOPERATIVE STUDY ON AREA BOMBING

By
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Jun 44

REPORT

SUBMITTED TO DIVISION 2
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By the

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Preface

This report is an extension of the results reached by Dr. Neyman and his group, and presented in an earlier report entitled "Cooperative Study on Probability of Exploding Land Mines by Bombing," which was prepared at the request of the Chairman of Committee DOLOC and distributed primarily to members of this Committee. The present report contains an analytical approach to the problem of area bombing by formations of planes which it is believed will be of interest to those concerned with the mathematical and statistical aspects of bombing accuracy and area bombing.

S U M M A R Y

I. The report deals with several problems arising in area bombing. The set of conditions common to all these problems is as follows.

A number of relatively small targets, termed sub-targets, idealized as circles of a given radius, or squares, are distributed over a rectangular area, termed target area. One or more formations, of N planes each, attack the target aiming the center of the bomb pattern at the center of the target area, with known standard errors of aiming for range and for deflection. The actual aiming is performed by the leading plane, all the others releasing their bombs on the leader. Each plane releases the same number of bombs, of the same type, in trains with the same spacing. The structure of the formations considered is subject to the following limitations. (a) The trains of all the planes participating in a formation have the same intended range, the deviations in this respect being due solely to the chance variations in the pattern flown and to the inaccuracies in timing the releases of single planes. (b) The lateral spacing of the planes does not exceed a certain specified limit (300' to 400', depending on the type of bomb and on the accuracy in flying the formation pattern), so that there are no noticeable crater free areas in the pattern of bombs.

It is obvious that, under the above conditions, the probability of a sub-target being hit depends on its location within the target area. It is found that those sub-targets most difficult to hit are located in the four corners of the target area.

Problem 1. Given the above conditions and the structure of the formations and the number of attacking formations, to determine the probability of a particular sub-target, of given location, being hit at least once.

Problem 2. Given the same conditions as in Problem 1, to determine the expected number of the sub-targets, located at random within the target area, which will be hit at least once.

Problem 3. Under the same conditions to determine (a) the expectation and (b) the standard error of the number of hits within any rectangle.

Problem 4. Having in view a knockout assault, to determine (a) the pattern of formation with which the chance of missing the sub-target in a corner of the target area (the one which is hardest to hit) is smallest , and (b) the number F_a of formations attacking with which this chance is reduced to the preassigned low level a.

Problem 5. Having in view routine bombing of many similar target areas, to determine the formation pattern maximizing the expected number of sub-targets randomly dispersed over the target area, which will be hit in a single attack.

The arithmetical and graphical procedures leading to the solution of the above problems are summarized in Section VII which includes also a few illustrations.

II. The solutions of the above problems are obtained on the basis of a new simplified idealization of formation bombing, based on the hypothesis of H. P. Robertson. It consists in regarding the ultimate distribution of bombs as due to two hypothetical actions. First the center of a rectangle, described as the bomb pattern, is aimed at the selected aiming point. Next, the bomb pattern being already placed, all the bombs dropped are randomly distributed over the rectangle.

Sections II to VI are given to the theory of the above hypothesis. In order to have an idea how well its consequences may agree with the actual facts, some of such consequences are compared with the experience of the VIII Bomber Command as described in the Report of Operations Analysis Section dated October 31st, 1943. Some other consequences of the theory, which could not be checked empirically for lack of data, are compared with the corresponding results of a more detailed, but much less manageable, theory of formation bombing which ignores fewer of its details.

Both methods of checking indicate that, with the radius of sub-targets not exceeding 100', the differences between the results of the theory of the Robertson hypothesis and the actual facts may be expected to be small and (as far as the practical conclusions are concerned that may be based on this theory) unimportant.

C O N T E N T S

CHARTS, GRAPHICAL AIDS TO COMPUTATION

I. PURPOSE

The present report is a sequel to the one previously submitted*, "Cooperative study on probability of exploding land mines by bombing." Its purpose is to present the results obtained to date on the extension of the previous results, corresponding to a greater number of variable parameters and to a wider range of variation of some of those considered previously, by utilizing certain approximations based on a new simplified idealization of the formation bombing, suggested by Dr. H. P. Robertson.

The First Cooperative Study contains nomograms for an easy computation of the probability P that a land mine placed in a corner of the proposed path 100' wide through the mine field (the mine which is the most difficult to destroy) will be missed by all the bombs dropped. The probability P is a function of several variables. In the nomograms, some of these variables were given values which seemed plausible in the expected conditions of bombing while the others were allowed to vary within certain limits. Thus the nomograms give the value of P corresponding to any system of values within these limits.

The arguments of P which were given fixed values are as follows.

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* In what follows this previous report will be referred to as the First Cooperative Study.

(1) The size and general shape of the formation flown, namely nine or eighteen plane formations flying in three or six successive "Vee's" of three planes each, with the spacings which appeared to be optimum for the purpose at hand.

(2) The standard errors of aiming for range and for deflection, $\sigma_{a_r} = \sigma_{a_d} = \sigma_a = 400'$. Some results obtained assuming $\sigma_a = 200'$ and $\sigma_a = 600'$ indicate that the standard error of aiming is an extremely important factor.

(3) The standard errors of dispersion of bombs in range and in deflection, $\sigma_{d_r} = 50'$, $\sigma_{d_d} = 30'$ respectively.

(4) The standard errors of intraformational dispersion in range and in deflection, $\sigma_{F_r} = 25'$, $\sigma_{F_d} = 10'$ respectively.

The variable arguments and the range of variation within which the nomograms determine the corresponding values of P, are as follows.

(i) The length B of the proposed path across the mine field, varying within the limits $400' \leq B \leq 1200'$

(ii) The radius of efficiency R of the bombs, with its range $6' \leq R \leq 18'$.

(iii) The number n of bombs released by each plane, $10 \leq n \leq 40$.

(iv) The number F of formations each aiming independently at the center of the proposed path across the mine field,

$1 \leqq F \leqq 60$.

The value of P was determined for a network of combinations of values of the parameters (i) to (iv) and also of an additional one, namely 2u = the spacing of bombs in train. Next the optimum spacing was determined and tabled and the nomograms give the values of P corresponding to arbitrary values of B, R, n and F and to the spacing of bombs which is optimum for these values.

The fixing of the arguments (1) to (4), thereby limiting the applicability of the results, is an obvious disadvantage. In the present report an attempt is made to provide an easy method of computing the approximate value of the probability P and also of certain other characteristics of the methods of bombing for varying values of these four parameters.

Apart from this it was found expedient to extend the applicability of the method to a wider range of the radius of efficiency R of bombs.

Original estimates of R with respect to land mines ranged from 6' to 18' for 100 lb. through 500 lb. bombs. Since the First Cooperative Study was submitted a report by Dr. Marston Morse and Captain Russel Baldwin (T.D.B.S. Report No. 27, March 23, 1944) was received by the authors indicating that if the 100 lb. G.P. bomb is fitted with an appropriate fuze then its original radius of efficiency of some 7' appears to increase to about 15'.

Therefore, it may be expected that, with a similar device, the radius of efficiency of a 500 lb. bomb may be over 30'. This is considerably outside the range of R in the nomogram of the First Cooperative Study.

In addition, it will be noticed that the nature of the problem of mine clearing is no different from the general problem of area bombing, in which a considerable number of planes release many bombs with only one plane aiming at a selected point, the other planes releasing on signal from the leader. In many such cases there is no single target. On the contrary, the area about the aiming point contains many sub-targets such as a factory and other buildings, parked planes, pillboxes, artillery and personnel, which perfectly correspond to land mines. The difference consists in the values of the parameters on which the probability of missing a sub-target depends and, in particular, on the value of the radius of efficiency of bombs.

The efficiency of missiles was studied by Marston Morse and William R. Transue. Their report (T.D.B.S. Report No. 28, April 3, 1944) gives tables of values of a number of characteristics as functions of the distance R from the point of explosion. The range of values of R for the 20 lb. fragmentation bomb M41 begins at R = 20' and extends into hundreds of feet. Thus R = 20' and R = 30' could hardly be considered exaggerated values for the radius of efficiency of the 20 lb. fragmentation bomb, at least

from some points of view. Comparing the performance of this bomb with others, one is led to believe that, in certain circumstances, the radius R of the 100 lb. G. P. and of the 500 lb. G. P. bombs must be of the order of 50' and 100' respectively. It follows that, with a wider range of problems of bombing in view, it is desirable to extend the range of values of R to 100' and, probably, even further.

The results obtained for the above conditions are summarized in graphs and nomograms which are believed to give data sufficiently accurate for many practical purposes and which are much more flexible than the nomograms given in the First Cooperative Study.

II. METHOD

More flexible nomograms of the type described could be produced by the method used in obtaining the results described in the preceding report. This, however, could not be achieved without considerable delay. Rapid results were made possible by adopting a new idealization of the problem suggested by a hypothesis of Dr. H. P. Robertson which was communicated to the authors by Dr. Warren Weaver as follows:

> "It is my understanding that Dr. H. P. Robertson has recently suggested that it would be reasonable to assume that 70 % of all bombs dropped from 11,000' altitude would be spread with approximate statistical regularity over a rectangle 1000'x 2000', the center of this rectangle having a CEP of 1000' with respect to the aiming point."

Essentially, a hypothesis of the same kind seems to be at the base of the Report of the Operations Analysis Section of the VIII Bomber Command dated October 31, 1943, "Analysis of VIII Bomber Command Operations from the Point of View of Bombing Accuracy", by Dr. W.J. Youden, Dr. J. A. Clarkson and Major P. C. Scott *.

The new idealization consists in regarding the process of formation bombing as satisfying the following two hypotheses. Let τ denote a rectangle of dimensions 2A x 2B, with the side 2B parallel to the direction of flight of the formation, and C the center of τ. This rectangle will be described as "pattern of bomb fall", or "bomb pattern".

> (i) The center C of the bomb pattern τ is aimed at the selected aiming point. The standard errors of aiming for range and for deflection are σ_{a_r} and σ_{a_d} respectively. It will be assumed that the errors of aiming X and Y for deflection and for range respectively are mutually independent and vary normally about zero.

Denote by N the number of planes in the formation and by n the number of bombs released by each plane, so that Nn represents the total number of bombs dropped by the formation. Let α

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* This Report will be quoted below as "VIII B.C. Report", for short.

be a number between zero and unity. The second part of the hypothesis refers to the distribution of the fraction α of all the bombs dropped, i.e. to the distribution of $m = \alpha Nn$ bombs, which, for distinctness, will be labelled "C-bombs", the letter C connoting "considered". If α is close to unity, then the number m of C-bombs will be close to that of all bombs dropped.

> (ii) Given the errors of aiming X, Y, the C-bombs are distributed over the pattern τ with statistical uniformity so that, if s is any area within τ, the probability of any one C-bomb hitting s is $\rho = s/4AB$ and the probability of there being exactly k hits by C-bombs within s is given by the binomial formula

$$\binom{m}{k}\rho^k(1-\rho)^{m-k} \tag{1}$$

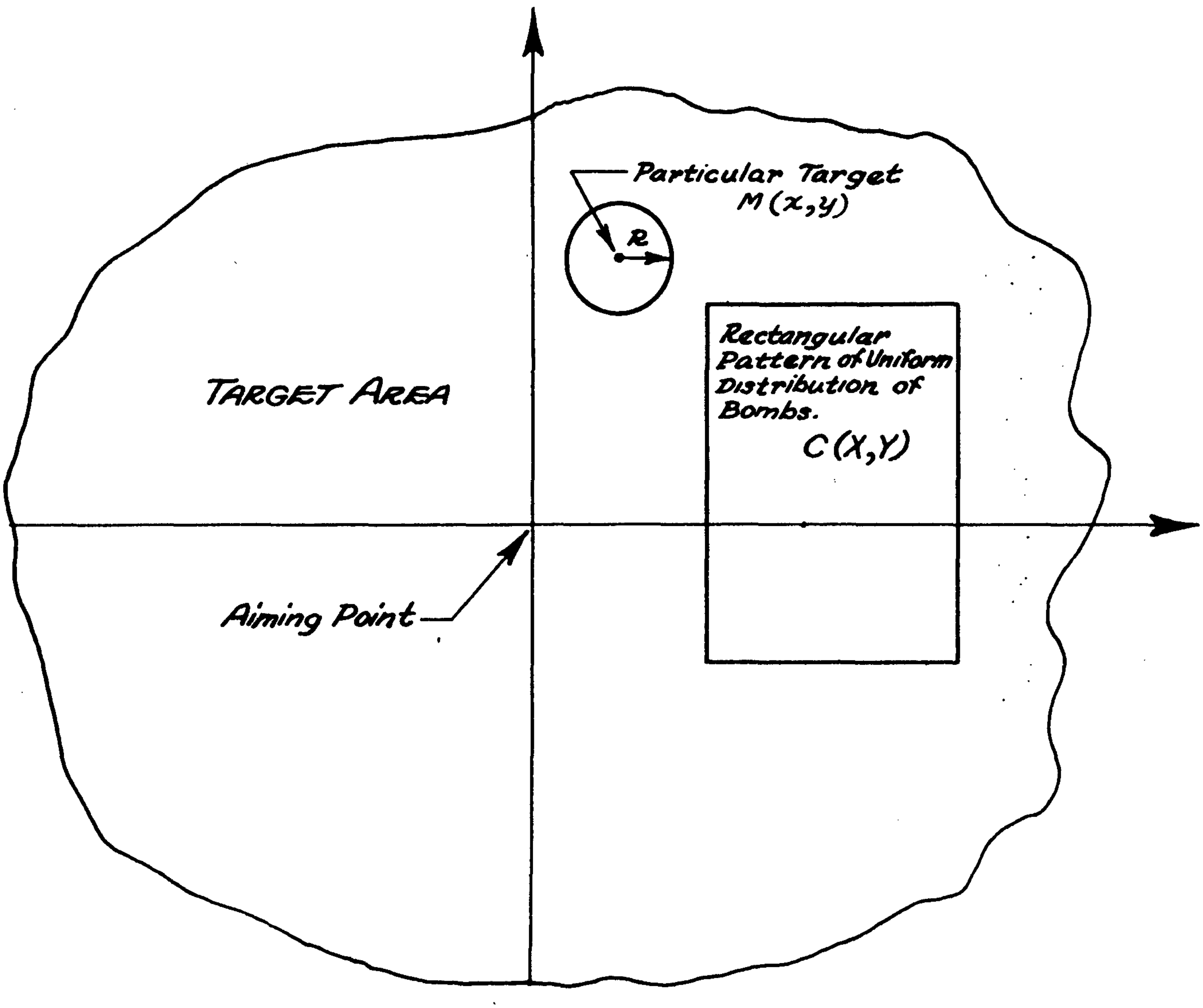
The situation is illustrated in Figure 1.

It will be seen that the applicability of the formulae obtained using the above hypotheses (i) and (ii) depends on the knowledge of (a) the dimensions 2A x 2B of the bomb pattern appropriate to the type of formation of planes contemplated for a given mission, (b) the number m of C-bombs corresponding to the number nN and (c) the standard errors of aiming σ_{ad} and σ_{ar}.

Although the elements under (a)and (b) may seem vague, the possibility of determining them was checked in a few cases where there were reliable data and the results (presented in further

Figure 1

General Situation



sections) proved to be surprisingly satisfactory. As a result, the authors feel that the probabilities computed on the basis of Robertson's* idealization of formation bombing are likely to be suitable for practical work.

III. PROBABILITY OF EXACTLY k HITS BY C-BOMBS.

Consider Figure 2. The axes of coordinates are drawn through the aiming point situated within a target area. The direction of the axis OY coincides with the direction of flight. The point (x,y) is the center of a particular target or "sub-target" such as a land mine, the center of a factory building, a tank, etc. The circle S(x,y) has its center at (x,y) and its radius equal to the radius of efficiency R of the bombs used.

Our problem consists in computing the probability $P_{F,k}$ that the C-bombs released will hit the circle S(x,y) exactly k times, as a result of F independent formation attacks. It is ob

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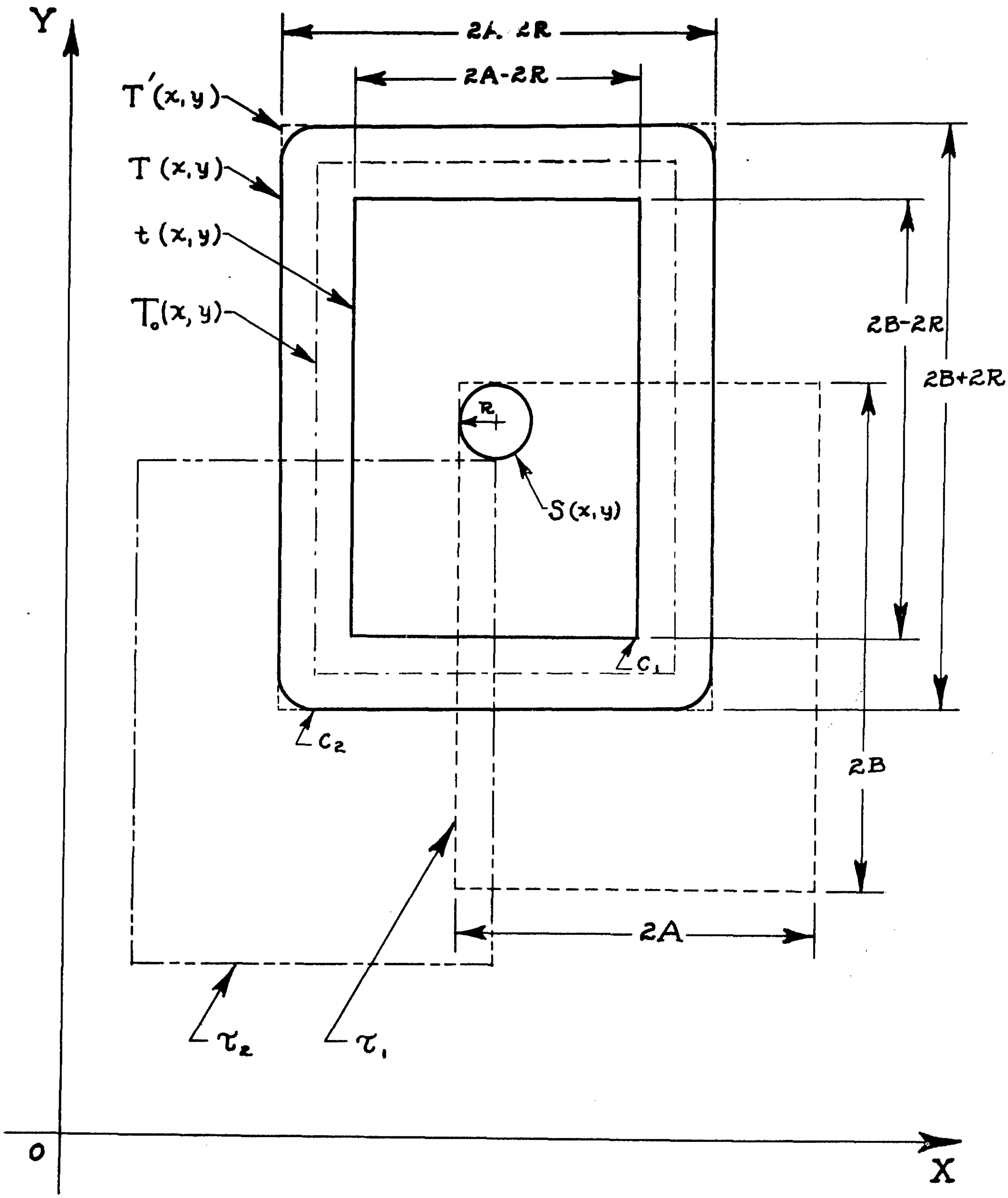
* The greater part of the present report is concerned with implications of the two assumptions (i) and (ii) which, essentially, constitute the hypothesis of Dr. Robertson. The authors feel that this hypothesis is likely to be very useful and significant and that, therefore, it is appropriate to label it with the name of its author. However, the authors have no indication as to whether or not Dr. Robertson would approve of the use made of his hypothesis in this Report. Therefore, when "Robertson's theory" or "Robertson's probabilities", etc. are mentioned, the authors wish to have it clearly understood that these and similar expressions are not meant to impose any responsibility on Dr. Robertson for possible misuse of his ideas.

— Figure 2 —



Regions $t(x,y)$, $T_0(x,y)$ and $T(x,y)$

vious that $P_{F,k}$ is a function of the following variables, x, y, m, R, A, B, σ_{a_d} and σ_{a_r}. However, for the sake of simplicity, these variables are not included in the symbol of the probability.

In order to compute $P_{1,k}$, draw a rectangle $T_o(x,y)$ with dimensions 2A x 2B, centered at (x,y), with its sides parallel to the axes. $T_o(x,y)$ would be the pattern of bombs if the errors of aiming X and Y were equal to x and y respectively. Denote by t(x,y) another rectangle centered at (x,y) inside of $T_o(x,y)$, with its sides parallel to those of $T_o(x,y)$ and distant from them by the quantity R. Thus the dimensions of t(x,y) will be 2(A-R) x 2(B-R).

Further, let T(x,y) be the area including $T_o(x,y)$ and limited by a contour representing the locus of points distant from the sides of $T_o(x,y)$ by the quantity R. Obviously, except for the corners cut by circles of radius R, the area T(x,y) is a rectangle, say T'(x,y), centered at (x,y) with its dimensions 2(A+R) x 2(B+R). Finally, the symbol (T-t) will stand for the band between the contours of T(x,y) and t(x,y).

It will be assumed that R < A,B. It will be seen that

(i) If the center C of the bomb pattern τ falls within t(x,y) then the bomb pattern τ entirely covers the circle S(x,y).

(ii) If the center C of the bomb pattern τ falls within (T-t), then the bomb pattern τ covers only a part of the circle S(x,y). The area of this covered part depends on the differences

$X-x$ and $Y-y$ between the errors of aiming and the coordinates of the center of $S(x,y)$, is easy to compute and will be denoted by $4AB \cdot f(X-x,Y-y)$.

(iii) If the center C of the bomb pattern τ falls outside of $T(x,y)$ then the pattern τ has no points in common with the circle $S(x,y)$ and, therefore, none of the C-bombs can hit $S(x,y)$.

Denoting by $P\{C\epsilon w\}$ the probability of the center of the bomb pattern falling within any specified area w, we have

$$P_{1,0} = 1-P\{C\epsilon T\} + (1-\rho)^m P\{C\epsilon t\} + \iint_{(T-t)} (1-f)^m p(X,Y)\,dX\,dY \tag{2}$$

where $\rho = \pi R^2/4AB$ represents the probability of one particular C-bomb hitting $S(x,y)$ if this circle is entirely covered by the bomb pattern, and where $p(X,Y)$ stands for the elementary probability law of the errors of aiming X and Y.

If $k > 0$, then the probability of exactly k hits by the C-bombs on the circle $S(x,y)$ is given by the formula

$$P_{1,k} = \binom{m}{k}\rho^k(1-\rho)^{m-k}P\{C\epsilon t\} + \binom{m}{k}\iint_{(T-t)} f^k(1-f)^{m-k}p(X,Y)\,dX\,dY \tag{3}$$

As is well known, whenever m is large, say $m > 25$, and ρ is small so that the product

$$D = m\rho = \frac{m\pi R^2}{4AB} \tag{4}$$

does not exceed a few units, then the binomial terms

$$\binom{m}{k}\rho^k(1-\rho)^{m-k} \tag{5}$$

differ but little from the Poisson limit

$$e^{-D}D^k/k! \tag{6}$$

With this approximation, and since $0 \leqq f \leqq \rho$, we may write, with $f_1 = mf$,

$$P_{1,0} = 1-P\{C\epsilon T\} + P\{C\epsilon t\}e^{-D} + \int\int_{(T-t)} e^{-f_1}p(X,Y)\,dX\,dY \tag{7}$$

$$P_{1,k} = \frac{1}{k!}\left\{P\{C\epsilon t\}e^{-D}D^k + \int\int_{(T-t)} e^{-f_1}f_1^k\; p(X,Y)\,dX\,dY\right\} \tag{8}$$

If the Poisson limit is a good approximation to the binomial, formulae (7) and (8) may be used to compute $P_{1,k}$ for $k = 0, 1, 2, \ldots$ Otherwise, formulae (2) and (3) may be used. Once the probabilities $P_{1,k}$ are computed, the values of the probabilities $P_{F,k}$ of exactly k hits on the circle S(x,y) by the C-bombs released by an arbitrary number F of formations will be found from the probability generating function

$$\Psi_F(z) = \left\{\sum_{r=0}^{m} P_{1,r}\cdot z^r\right\}^F \tag{9}$$

namely

$$P_{F,k} = \frac{1}{k!}\left.\frac{d^k\Psi}{dz^k}\right|_{z=0} \qquad k = 0, 1, 2, \ldots \tag{10}$$

In particular,

$$
\begin{aligned}
P_{F,0} &= P_{1,0}^{F} \\
P_{F,1} &= F \cdot P_{1,0}^{F-1} \cdot P_{1,1} \\
P_{F,2} &= F \cdot P_{1,0}^{F-1} \cdot P_{1,2} + \tfrac{1}{2} F(F-1) \cdot P_{1,0}^{F-2} \cdot P_{1,1}^{2} \\
P_{F,3} &= F \cdot P_{1,0}^{F-1} \cdot P_{1,3} + F(F-1) \cdot P_{1,0}^{F-2} \cdot P_{1,1} \cdot P_{1,2} \\
&\quad + \frac{1}{3!} F(F-1)(F-2) \cdot P_{1,0}^{F-3} \cdot P_{1,1}^{3}
\end{aligned}
\tag{11}
$$

etc.

Although the computation of formulae (2), (3), (7), and (8) is straightforward, it is somewhat time consuming because of the presence of the integral involving the function $f(X-x,Y-y)$. As the latter is bounded by zero and ρ, the following bounds are obtained for $P_{F,0}$.

$$\left(1-P\{C\epsilon T\}[1-(1-\rho)^{m}]\right)^{F} < P_{F,0} < \left(1-P\{C\epsilon t\}[1-(1-\rho)^{m}]\right)^{F} \tag{12}$$

Since the difference between the regions $t(x,y)$ and $T(x,y)$ depends on the value of R, with the region $T_0(x,y)$ being intermediate between $t(x,y)$ and $T(x,y)$, it is concluded that, in cases where R is small compared with A, B, σ_{a_d} and σ_{a_r}, a reasonable approximation to $P_{F,0}$ may be obtained by using either the expression

$$P^{*}_{F,0} = \left[1 - P\{C\epsilon T_0\}(1-e^{-D})\right]^{F} \tag{13}$$

or the expression

$$P^{**}_{F,0} = \left[1 - P\{C\epsilon T_0\}\Big(1-(1-\rho)^m\Big)\right]^F \tag{14}$$

The use of either (13) or (14) instead of the accurate values of $P_{F,0}$ obtained either from (7) or from (2) is equivalent to approximating the function f by a step function, say $f_1 = \rho$ within the region $T_0(x,y)$ and $f_1 = 0$ elsewhere. Using this approximation throughout and denoting $P\{C\epsilon T_0\}$ by I, we obtain,

$$P^{*}_{1,k} = I\cdot e^{-D}D^k/k! \tag{15}$$

$$P^{**}_{1,k} = \binom{m}{k} I\cdot\rho^{k}\cdot(1-\rho)^{m-k} \tag{16}$$

Putting for convenience

$$g(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2} \tag{17}$$

$$G(x) = \int_0^x g(t)dt \tag{18}$$

the value of $I = P\{C\epsilon T_0\}$ is found to be

$$I = P\{C\epsilon T_0\} = \left(G\left(\frac{x+A}{\sigma_{a_d}}\right) - G\left(\frac{x-A}{\sigma_{a_d}}\right)\right)\left(G\left(\frac{y+B}{\sigma_{a_r}}\right) - G\left(\frac{y-B}{\sigma_{a_r}}\right)\right) \tag{19}$$

As mentioned, the approximation provided by (13), (14), (15) and (16) depends upon the value of R. The closeness of the approximation may be judged from Table I, which relates to two different sets of conditions with values of R alternatively 60' and 100'.

TABLE I

Comparison of Accurate Values of $P_{F,k}$ with

Approximations $P^*_{F,k}$

Conditions	R=60', F=10		R=100', F=6	
k	$P_{F,k}$	$P^*_{F,k}$	$P_{F,k}$	$P^*_{F,k}$
0	.060	.066	.067	.093
1	.102	.105	.041	.032
2	.137	.140	.060	.059
3	.149	.151	.076	.077
4	.140	.142	.083	.085
5	.156	.155	.086	.085

Having reference to 500 lb. bombs with a maximum load of n = 12, and formations of N = 18 planes, Table I assumes that m = 216, $\alpha = 1$. Values of the other constants are those suggested by Robertson, 2A = 1000', 2B = 2000', $\sigma_{a_d} = \sigma_{a_r} = 849.3'$. The coordinates of this particular target are x = y = 0. It will be seen that the agreement between the exact values of $P_{F,k}$ based on formulae (7) and (8) and the approximation $P^*_{F,k}$ of formulae (13) and (15) is satisfactory when R = 60'. With R = 100' the agreement between $P_{F,k}$ and $P^*_{F,k}$ is again excellent for $k \geqq 2$. For k = 0, which is probably the most important value of k, the difference between the true and the approximate value of the probability is rather large, equal to .026. However, even with

this large deviation, it could scarcely be denied that $P^*_{F,k}$ preserves the order of magnitude of the true probabilities of exactly k hits on the circle S(x,y).

As a result of these and similar computations it is assumed that formulae (13) and (15) are likely to give reasonably accurate values of the probabilities $P_{F,k}$, for $R \leqq 60'$, and probably also for $R < 100'$. For greater values of R more accurate computations would be desirable.

Having in view the restricted range of values of $R < 100'$, a chart and a nomogram were constructed for an easy computation of the approximate value $P^*_{F,0}$ of the probability $P_{F,0}$ that none of the C-bombs released by F formations will hit the circle S(x,y) of radius R, centered at a point (x,y). The description of the chart and the nomogram and some examples of their use will be found at the end of this report.

It will be noticed that the probabilities $P^*_{F,k}$ fall into the category of "contagious" distributions recently studied by one of the authors and W. Feller*.

IV. COMPARISON WITH THE POISSON LAW

It is currently understood that, in order to estimate the probability of exactly (or at least) k explosions of bombs

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*Ann. Math. Stat., Vol. X (1939) p. 35 and Vol. XIV (1943) p. 389.

within a given circle S around some particular target within the general target area, attempts are made to use the Poisson Law, namely,

$$P_k' = e^{-\lambda}\frac{\lambda^k}{k!} \qquad (20)$$

with λ denoting the average density of bombs per area of the circle S. Therefore, it seems useful to inquire under what conditions the probabilities $P^*_{F,k}$, considered as a function of k, conform with the law (20). Using (13) and (15) the probability generating function of k is found to be

$$\Psi_F(z) = \{1 - I + I \cdot e^{-D}e^{Dz}\}^F \qquad (21)$$

where, for simplicity of writing,

$$I = P\{C \epsilon T_o\}. \qquad (22)$$

The probability generating function of the Poisson Law (20) is

$$\varphi(z) = e^{-\lambda(1-z)}. \qquad (23)$$

It will be seen that, for the identity of (21) and (23), it is both necessary and sufficient that

$$1 - I + I \cdot e^{-D} = e^{-\lambda/F} \qquad (24)$$

$$I \cdot e^{-D}D^k = e^{-\lambda/F}(\lambda/F)^k \quad k = 1,2,... \qquad (25)$$

and, therefore,

$$\lambda = FD, \ I = 1. \qquad (26)$$

Thus, I = 1 is the only case in which the probabilities $P^{*}_{F,k}$ are identical with those given by the Poisson Law. This case arises when the dimensions 2A x 2B of the bomb pattern τ are greater than a certain multiple of the standard errors of aiming and when the particular sub-target considered is close to the aiming point. In cases of this kind the bomb pattern τ covers the whole circle S(x,y) practically always and, consequently, the ultimate distribution of bombs within and around S(x,y) is statistically uniform, whatever be F.

Other conditions under which $P^{*}_{F,k}$ may approach the Poisson formula are as follows. Let D tend to zero and F increase so that the product FD remains constant, say FD = ν. Then the probability generating function $\Psi_F(z)$ tends to the limit

$$\text{Lim } \{1 - I + I \cdot e^{-D(1-z)}\}^{\frac{\nu}{D}} = e^{-\nu I(1-z)} \tag{27}$$

$$= e^{-FDI(1-z)}$$

This is the probability generating function of the Poisson Law (20) with its mean

$$\lambda = FDI \tag{28}$$

Thus it may be stated that the second case of the probabilities $P^{*}_{F,k}$ being approximately equal to those determined by the Poisson Law arises when

$$D = \frac{m \pi R^2}{4AB} \tag{29}$$

is extremely small, while F is large, so that the product FD is moderate.

The expression of the probability generating function (21) may be used to determine the expected number of hits

$$\bar{k} = E(k) = FDI \tag{30}$$

and the variance of k

$$\sigma_k^2 = FDI\{1 + D(1-I)\} \tag{31}$$

Since with the Poisson Law the variance is exactly equal to the mean it is interesting to note that, unless $D = 0$ and/or $I = 1$,

$$\bar{k} < \sigma_k^2. \tag{32}$$

For a numerical comparison of the probabilities $P^*_{F,k}$ with those determined by thePoisson Law having the same mean $\lambda = FDI$, it is important to know the range of combinations of values of I and D which are likely to be met in practice. These values depend on x, y, m, R, A, B, and on the standard errors of aiming.

Since reports from the war theaters frequently mention standard errors of aiming of order 1000', this value is assumed in the numerical data which follow. In selecting values for the dimensions of the bomb pattern a distinction is made as to the pattern of formations flown. In some theaters three string formations are used and the dimensions of the bomb pattern obtained

in experimental bombing with these formations (see Figures 5,6) are of the order of 400' x 1500'. Another system of values considered is that suggested by Dr. Robertson, 1000' x 2000'. The third system considered is that reported as "average" in the VIII B. C. Report, p. 23, namely 2350' x 3500'. These three systems of pattern dimensions are combined with different values of m and R, depending on the purpose of bombing and on the loading capacity of the planes. As mentioned before, the radius of efficiency with respect to land mines is of the order of 15' for the modified 100 lb. bomb. As the radius R varies proportionately to the square root of the weight of the bomb (reports of the Land Mines Sub-Committee of the Advisory Council), it may be expected that with 250 lb. and 500 lb. bombs the radii will be about $15\sqrt{2.5}$ and $15\sqrt{5}$ feet respectively. For area bombing whose purpose is other than the clearing of mines, the radius R may be as large as 100'.

With this is mind, the four groups of hypothetical conditions given in Table II were formed. Two of the groups refer specifically to the problem of clearing mine fields and the other two refer to general purpose bombing.

The comparison of probabilities $P^{*}_{F,k}$ with probabilities P'_k determined by the Poisson Law having the same mean λ = FDI is given in Table III for a number of combinations of the three arguments, covering the range exhibited in Table II.

TABLE II

Plausible Values of I and D

PART 1: PROBLEM OF CLEARING MINE FIELDS

Bomb	R	n	Values of I and D with $\alpha = .9$, A = 200', B = 750', I(0,0) = .0867	Values of I and D with $\alpha = .7$, A = 500', B = 1000', I(0,0) = .2614
100 lb.	15'	40	.763	.178
250 lb.	$15\sqrt{2.5}'$	16	.763	.178
500 lb.	$15\sqrt{5}'$	12	1.145	.267

PART 2: GENERAL PURPOSE BOMBING

Bomb	R	n	Values of I and D with $\alpha = .7$, A = 500', B = 1000', I(0,0) = .2614	Values of I and D with $\alpha = .9$, A = 1175', B = 1775', I(0,0) = .7023, I(1000,0) = .5125
20 lb. Frag.	30'	144	2.565	.791
100 lb.	60'	40	2.850	.878
500 lb.	100'	12	2.375	.732

TABLE III

Comparison of Probabilities $P^{*}_{F,k}$ and P'_k

k	I = .2614, F= 20, D=.15: $P^{*}_{F,k}$	FDI=.7842: P'_k	I = .2614, F= 20, D=.25: $P^{*}_{F,k}$	F= 10, D=.50: $P^{*}_{F,k}$	FDI=1.307: P'_k
0	.476	.456	.304	.338	.271
1	.334	.358	.328	.298	.354
2	.136	.140	.210	.193	.231
3	.041	.037	.100	.100	.101
4	.010	.007	.039	.044	.033
5	.002	.001	.014	.017	.009

k	I = .2614, F= 10, D=.75: $P^{*}_{F,k}$	F= 5, D=1.50: $P^{*}_{F,k}$	FDI=1.961: P'_k	I = .2614, F= 10, D=1.00: $P^{*}_{F,k}$	F= 5, D=2.00: $P^{*}_{F,k}$	FDI=2.614: P'_k
0	.227	.312	.141	.164	.278	.073
1	.244	.176	.276	.158	.127	.191
2	.209	.171	.271	.177	.150	.250
3	.145	.128	.177	.155	.133	.218
4	.087	.086	.087	.115	.103	.142
5	.047	.054	.034	.077	.074	.074

k	I = .2614, F= 5, D=2.50: $P^{*}_{F,k}$	FDI=3.268: P'_k	I = .2614, F= 5, D=3.00: $P^{*}_{F,k}$	FDI=3.921: P'_k
0	.254	.038	.240	.020
1	.090	.124	.062	.078
2	.124	.203	.100	.152
3	.126	.222	.113	.199
4	.108	.181	.106	.195
5	.086	.118	.090	.153

TABLE III - continued

Comparison of Probabilities $P^*_{F,k}$ and P'_k

	I=.0867 F= 20 D=.75	FDI=1.300	I=.0867 F= 20 D=1.00	FDI=1.733	I=.0867 F= 20 D=1.25	FDI=2.167
k	$P^*_{F,k}$	P'_k	$P^*_{F,k}$	P'_k	$P^*_{F,k}$	P'_k
0	.392	.272	.324	.177	.279	.115
1	.252	.354	.219	.306	.185	.248
2	.172	.230	.179	.265	.173	.269
3	.096	.100	.121	.153	.132	.194
4	.048	.032	.073	.066	.091	.105
5	.023	.008	.041	.023	.059	.046

TABLE III - concluded

Comparison of Probabilities $P^*_{F,k}$ and P'_k

k	I=.7023 F= 5 D=.75	FDI=2.634	I=.5125 F= 5 D=.75	FDI=1.922
	$P^*_{F,k}$	P'_k	$P^*_{F,k}$	P'_k
0	.099	.072	.207	.146
1	.195	.189	.257	.281
2	.228	.249	.224	.270
3	.195	.219	.152	.173
4	.135	.144	.086	.083
5	.079	.076	.043	.032

	I=.7023 F= 5 D=1.00	FDI=3.512	I=.5125 F= 5 D=1.00	FDI=2.563
k	$P^*_{F,k}$	P'_k	$P^*_{F,k}$	P'_k
0	.053	.030	.141	.077
1	.124	.105	.197	.198
2	.176	.184	.208	.253
3	.189	.215	.173	.216
4	.164	.189	.122	.139
5	.122	.133	.076	.071

	I=.7023 F= 5 D=1.25	FDI=4.390	I=.5125 F= 5 D=1.25	FDI=3.203
k	$P^*_{F,k}$	P'_k	$P^*_{F,k}$	P'_k
0	.031	.012	.103	.041
1	.078	.054	.149	.130
2	.127	.120	.179	.208
3	.158	.175	.171	.223
4	.162	.192	.125	.178
5	.143	.168	.103	.114

Figure 3 gives six diagrams illustrating graphically some of the above distributions. A few points should be mentioned.

(i) As long as the product FDI = λ remains constant the Poisson Law remains unchanged, even though the particular parameters vary. On the other hand, the values of $P^{*}_{F,k}$ may vary strongly.

(ii) The value of $P^{*}_{F,0}$ is always greater than the corresponding value of P'_{0}. This is a consequence of a general theorem concerning contagious distributions, recently proved by W. Feller. In other words, the estimate of $P^{*}_{F,0}$ provided by the Poisson Law with the same mean is always optimistic. Occasionally it is a very optimistic estimate.

(iii) Graphs 2 and 3 in Figure 3 correspond to the same bombing conditions with one exception. Graph 2 refers to a sub-target placed exactly at the aiming point, so that x = y = 0. On the other hand graph 3 refers to the sub-target with coordinates x = 1000', y = 0. Since target areas roughly filling up a circle of about 1000' radius appear frequently as bombing objectives, these two graphs give an idea of the variation in the distribution of bombs within such a circle. It will be noticed that in the particular example illustrated the ratio of the two probabilities $P^{*}_{F,0}$ is about 1 to 2.

(iv) The graphs illustrate the theoretical conclusion

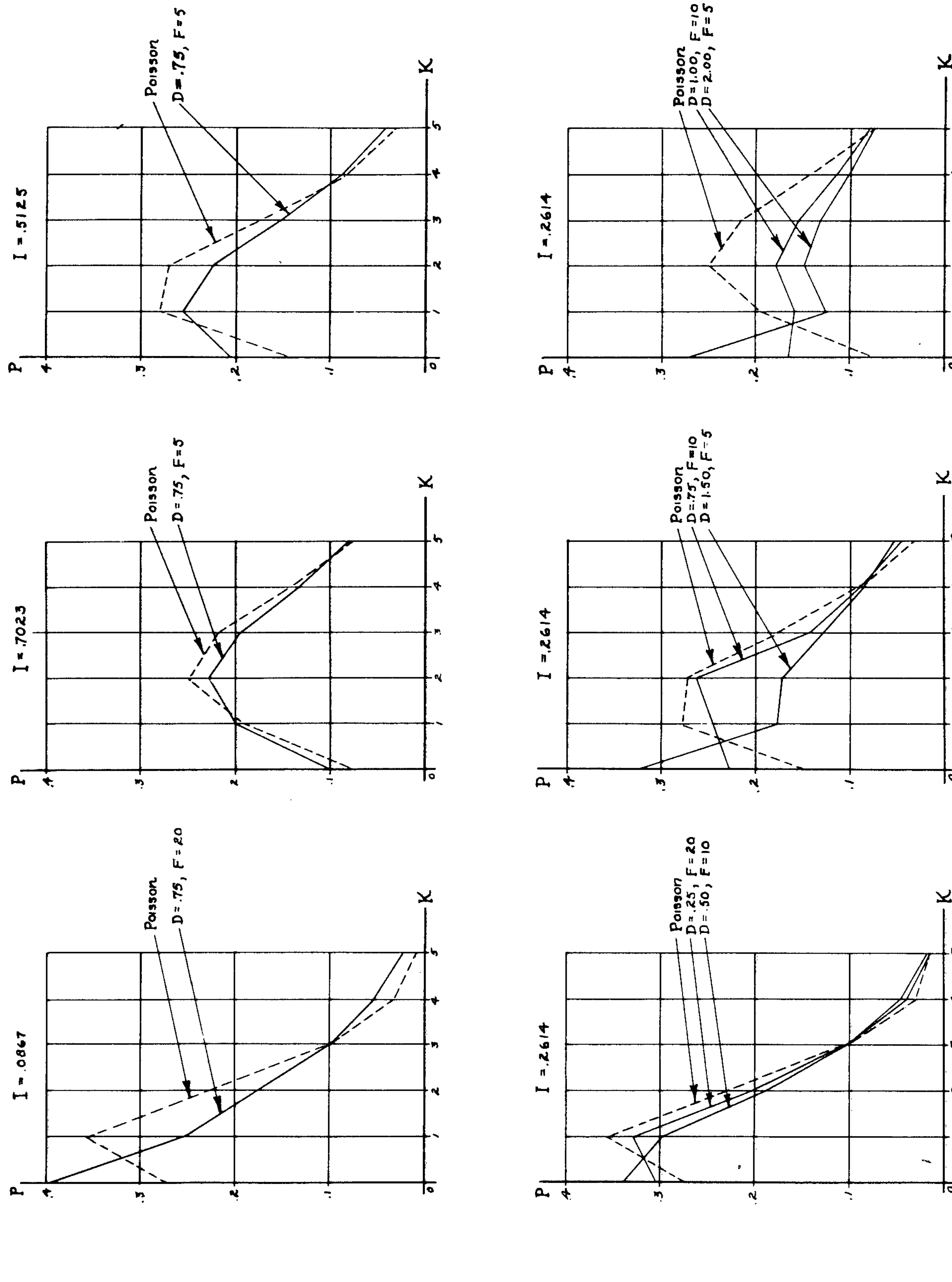

Figure 3

Probabilities $P^*_{F,0}$ and P'_K

reached above, that, when either D is small or I is large, the values of $P^{*}_{F,k}$ approach those determined by the Poisson Law.

V. VERIFICATION OF THE APPLICABILITY OF THE THEORY OF THE ROBERTSON HYPOTHESIS.

A. General. Figures 4, 5 and 6 reproduce bomb plots obtained in actual bombing. Figure 4 gives the plot published in VIII B.C. Report, p. 11, as a typical bomb pattern with the rectangle fitted to it in accordance with the practice of the VIII Bomber Command. The somewhat more distinct bomb plots given in the other two figures were obtained from photographs of experimental bombing. The originals of these photographs were kindly supplied by Dr. Walker Bleakney of Division 2, NDRC.

All three plots indicate the following difficulties which may be involved in applications of the theory of Robertson's hypothesis.

(i) The rectangular pattern within which the C-bombs are supposed to be uniformly and independently distributed, is not a reality in the same sense in which a formation of, say, 18 planes trying to fly in combat box stagger pattern is a reality.

There are two consequences of this fact which must be considered. First, given a most detailed description of the intended pattern of the formation, of the number and spacing of bombs and of the ability of the crews expressed in terms of the various standard errors, it is not known what values of the di-

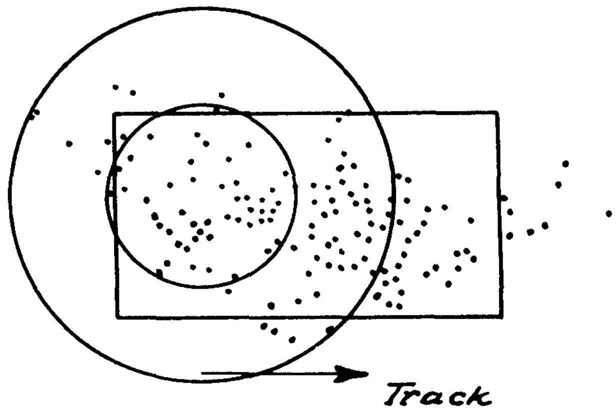

Figure 4

Typical Bomb Plot, VIII B. C.

(Taken from Operations Analysis Report, ANALYSIS OF VIII BOMBER COMMAND OPERATIONS FROM THE POINT OF VIEW OF BOMBING ACCURACY, October 31, 1943, p. 11)

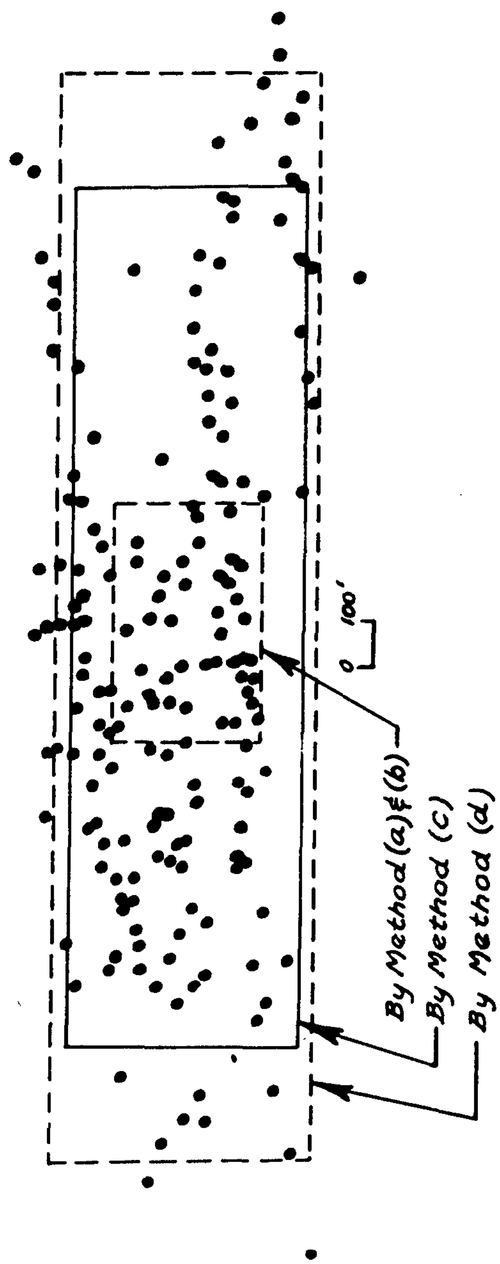

Figure 5

Bomb Plot. Experimental Bombing.

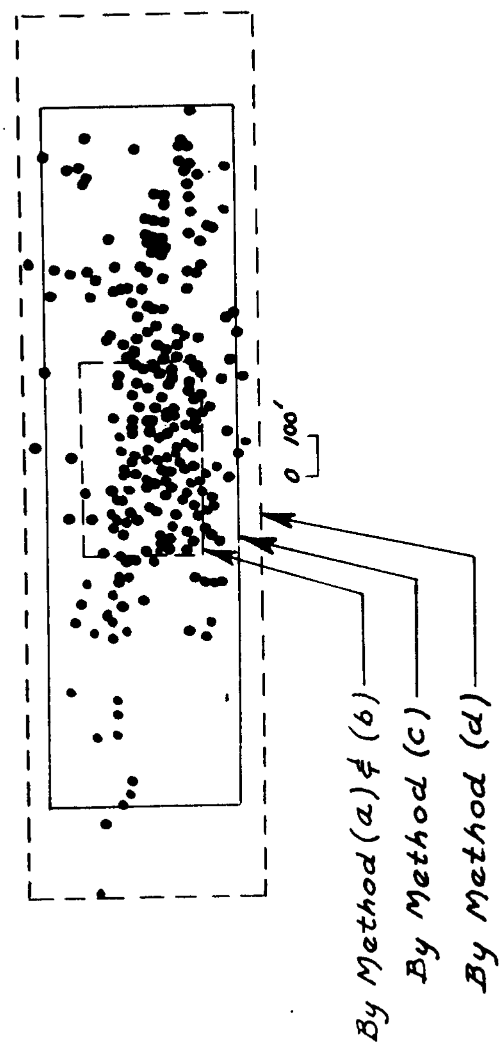

Figure 6

Bomb Plot. Experimental Bombing.

mensions 2A x 2B of the bomb pattern all these conditions would imply. Second, given typical plots of the bomb fall, ordinarily produced by formations of a specified type, such as those given in Figures 4 to 6, two different persons attempting to determine the dimensions of the rectangular bomb pattern are very likely to obtain discordant results unless some definite rule is followed. Also, even if these results were in agreement, it is not apparent that, substituted into the formulae for $P^*_{F,k}$, they will give results conforming with observation.

(ii) Uncertainty about the pattern dimensions implies uncertainty of the proportion α of all the bombs dropped which should be considered as uniformly distributed within the pattern.

Having in view the above difficulties, the next three subsections of this report discuss the following two questions:

(a) Whether or not the values of the pattern dimensions obtained by the Analysts of the VIII Bomber Command, when substituted into the formulae of the theory of Robertson's hypothesis, lead to results conforming with other empirical data published in the VIII B. C. Report;

(b) Whether or not the pattern dimensions 2A x 2B could be determined from some specific features of the formation and method of bombing, so that the probabilities computed on Robertson's hypothesis would agree with those computed on the more detailed theory of bombing outlined in the First Cooperative Study.

B. <u>Verification of (a)</u>. The first five columns of Table IV reproduce the data printed in the VIII B. C. Report on page 23. The dimensions of the bomb patterns are averages of a number of observations. The last column of Table IV gives the expected percent of bombs within a square whose center is the aiming point and whose area is equal to that of a 1000' circle. These figures were computed on the basis of Robertson's hypothesis, using the formulae deduced in Section C. The computations refer to a square rather than to a circle because it is believed that the difference between the two expectations is too small to justify the rather complex computations refering to a circle. For a square (or for any rectangle with its sides parallel to the axes of coordinates) the formula giving the expected proportion of hits is very simple. It will be seen that the order of magnitude of figures representing the expected and the observed bomb fall is distinctly the same, the greatest discrepancy corresponding to the largest bomb pattern.

As each of the lines of Table IV refers to a number of attacks which are likely to have differed considerably in the patterns of bombs, it is possible that some of the discrepancy in the last line may be due to variability in pattern dimensions of the missions to which the data refer. Therefore it was decided to make other computations on the raw data published at the end of the VIII B. C. Report.

TABLE IV

Bomb Patterns and the Average Percentage of Bombs Within Stated Areas about the Aiming Point. Source: VIII B.C. Report p.23

2A	2B	σ_a	Number of Observations	Average Percent Fall: Actually observed within a 1000' circle	Average Percent Fall: Computed from Robertson's hypothesis for a square of area $(1000')^2 \times \pi$
2050'	2500'	878'	64	35.0	32.3
2350'	3550'	1013'	62	23.8	22.9
3200'	4800'	997'	57	20.2	16.6

The particular missions to which the published data refer are identified in the VIII B. C. Report, Appendix I, by the location of the target and the date of the attack. For the sake of brevity and convenience in reference the missions were numbered in order, 1, 2, 3, etc. Next a selection was made of the pattern dimensions which appeared to be most frequent. The most frequent patterns, measured in 100', are 20 x 25, 20 x 30, 20 x 35, 25 x 25, and 25 x 30. The missions corresponding to these patterns are listed below in Tables V through IX, with all the data which are relevant for the purpose at hand, namely

TABLE V

Number of Bombs Dropped, Percent Falling Within Stated Areas about Aiming Point and the Errors of Aiming in Missions of the VIII Bomber Command. Source: VIII Bomber Command Report dated 31 October 1943, Appendix I.

Dimensions of the bomb pattern 2000' x 2500'

Mission Number	nN	Percent Within 1000'	Percent Within 2000'	X	Y
20	140	60	100	- 5	2
22	75	0	53	18	3
57	180	--	--	--	--
62	40	4	38	21	- 4
142	180	44	100	7	- 7
153	128	44	94	2	2
169	108	63	93	3	- 2
196	288	20	45	2	12
235	90	38	92	11	- 2
237	85	0	12	16	18
250	170	53	100	7	- 1
282	150	0	0	10	-38
304	168	21	68	-11	- 4
351	200	0	0	41	-29*
363	180	34	84	8	- 7
405	34	0	0	48	-82*
428	216	3	43	-18	- 8
454	252	62	98	1	- 5
485	42	65	100	4	- 5
543	140	6	71	14	4
639	95	0	0	-36	-78*

TABLE VI

Number of Bombs Dropped, Percent Falling Within Stated Areas about Aiming Point and the Errors of Aiming in Missions of the VIII Bomber Command. Source: VIII B.C. Report dated 31 October 1943, Appendix I.

Dimensions of the bomb pattern 2000' x 3000'

Mission Number	nN	Percent Within 1000'	Percent Within 2000'	X	Y
35	160	31	71	5	7
49	108	--	--	--	--
64	26	29	94	- 8	6
120	38	0	0	13	-23
137	320	57	78	- 1	- 2
240	32	21	67	-11	-10
329	288	17	24	0	-15
344	136	31	69	- 2	- 9
345	95	44	96	1	- 6
348	190	0	0	-44	76 *
361	192	0	0	0	-43
370	144	--	--	-54	31 *
389	120	28	79	10	3
426	75	0	0	32	-18
452	240	51	87	7	- 1
519	228	0	0	--	--
536	32	10	76	13	4
545	252	32	83	- 8	- 4
588	252	68	100	5	0
606	480	45	94	- 6	5

TABLE VII

Number of Bombs Dropped, Percent Falling Within Stated Areas about Aiming Point and the Errors of Aiming in Missions of the VIII Bomber Command. Source: VIII B.C. Report dated 31 October 1943, Appendix I.

Dimensions of the bomb pattern 2000' x 3500'

Mission Number	nN	Percent Within 1000'	Percent Within 2000'	X	Y
8	75	--	--	-84	-10 *
161	70	40	74	- 1	- 7
174	85	3	55	18	- 3
186	32	45	93	0	- 4
219	90	28	40	5	- 2
284	38	10	62	-13	- 8
306	156	24	65	5	7
422	85	30	60	3	-12
479	252	29	81	10	4
480	252	0	20	23	6
515	276	46	99	1	5
567	480	42	82	3	- 4
585	252	43	99	5	0
587	252	0	12	25	4

TABLE VIII

Number of Bombs Dropped, Percent Falling Within Stated Areas about Aiming Point and the Errors of Aiming in Missions of the VIII Bomber Command. Source: VIII B.C. Report dated 31 October 1943, Appendix I.

Dimensions of the bomb pattern 2500' x 2500'

Mission Number	nN	Percent Within 1000'	Percent Within 2000'	X	Y
151	144	30	83	- 7	- 8
178	70	0	0	34	8
322	190	10	50	- 2	-20
324	200	8	51	-10	-16
365	238	0	2	- 9	-24
372	128	57	100	5	0
523	228	26	93	0	8
595	216	0	8	- 2	24
607	190	58	94	0	1
614	252	21	72	13	5
620	204	32	73	-10	5
621	95	39	85	- 4	- 3
624	42	5	40	20	2

TABLE IX

Number of Bombs Dropped, Percent Falling Within Stated Areas about Aiming Point and the Errors of Aiming in Missions of the VIII Bomber Command. Source: VIII B.C. Report dated 31 October 1943, Appendix I.

Dimensions of the bomb pattern 2500' x 3000'

Mission Number	nN	Percent Within 1000'	Percent Within 2000'	X	Y
2	100	30	62	- 8	- 2
17	120	3	50	-15	5
36	432	0	0	28	- 6
68	14	6	24	22	-13
104	120	53	76	4	7
175	160	46	89	3	3
176	200	36	73	4	14
180	160	45	98	4	- 8
323	210	19	55	13	- 7
340	192	0	0	-59	16 *
367	132	52	99	0	0
408	170	0	0	0	-52 *
409	110	5	26	- 6	-21
414	28	0	0	6	-32
447	190	7	30	4	-24
461	798	37	86	- 4	- 6
551	36	11	44	12	-12
572	240	17	73	14	- 5
605	190	0	0	38	90 *

identification number, number of bombs dropped, percent of bomb fall within the 1000' and the 2000' circles and the aiming errors in deflection and range.

Following the established practice, the data were worked twice. First, all of the missions of each category were used to estimate the standard errors of aiming and for all the other computations. Next an attempt was made to eliminate missions in which a "gross error" was likely to have occurred. It was assumed arbitrarily that a circular error of aiming of 5000' or more marks a "gross error". Missions which were eliminated on this criterion are marked by asterisks. Table X summarizes the results obtained in this way, first with respect to all the missions and then with respect to the data cleaned of gross errors.

Columns 3 to 6 give the observed and the expected average percent of bomb fall within the stated areas supplemented by $\pm$ the standard error of the average in question. The purpose of this supplement is twofold. First, the theoretical value of the S.E. is useful when judging the discrepancy between the expected and the observed percentages. Second, the comparison of the observed with the expected S.E. gives an idea of whether the actual variability of the percent of bombs within a given area, from one mission to another, corresponds to the expectation based on Robertson's scheme.

TABLE X

Pattern Dimensions, Percentage of Bomb Fall Within Stated Areas About Aiming Point and the Frequency of Total Misses. Source: VIII Bomber Command Report dated 31 October 1943, Appendix I.

PART 1: ALL OBSERVATIONS

Pattern in 100'	σ_{a_d} σ_{a_r}	Average Percent of Bombs ± S.E.				Frequency of Total Misses			
		Observed within 1000' circle	Computed for a square of area $\pi 10^6$	Observed within 2000' circle	Computed for a square of area $4\pi 10^6$	Obs. for 1000' circle	Exp. L. bound for sq. of area $\pi 10^6$	Obs. for 2000' circle	Exp. L. bound for sq. of area $4\pi 10^6$
20 x 25	1830' 2634'	25.8± 5.7	9.1± 3.5	59.6± 8.8	31.1± 7.6	6	11.8	4	7.0
20 x 30	1967' 2430'	25.8± 5.1	9.0± 3.4	56.6± 9.3	31.1± 9.3	5	10.0	5	5.6
20 x 35	2616' 622'	26.2± 4.7	12.8± 4.2	64.8± 7.7	42.5± 9.7	2	6.6	0	3.8
25 x 25	1305' 1303'	22.0± 5.7	19.7± 4.7	57.8±10.0	58.2± 8.4	3	2.5	1	0.5
25 x 30	1966' 2732'	19.3± 4.5	7.9± 2.9	46.6± 8.3	31.9± 6.0	5	10.5	5	6.2

PART 2: MISSIONS WITHOUT "GROSS ERRORS"

20 x 25	1008' 1144'	30.4± 6.1	26.5± 4.7	60.1± 7.9	69.8± 6.3	3	2.0	1	0.2
20 x 30	1063' 1316'	27.3± 5.3	22.4± 4.2	63.5± 9.0	63.0± 7.0	4	2.2	4	0.4
20 x 35	1041' 598'	26.2± 4.9	26.7± 3.9	64.8± 7.7	75.1± 6.0	2	0.9	0	0.1
25 x 25	1305' 1303'	22.0± 5.7	19.7± 4.7	57.8±10.0	58.2± 8.4	3	2.5	1	0.5
25 x 30	1101' 1192'	22.9± 4.9	21.9± 3.6	55.3± 8.1	62.8± 6.5	2	1.5	2	0.2

The last four columns refer to another important characteristic of the bombing methods, namely the number of cases in which all of the bombs dropped miss either the 1000' or the 2000' circle (or the corresponding squares) about the aiming point. No accurate and easy theoretical formula is now available to compute the expectation of this quantity. On the other hand Table X gives the values of a lower bound of the expectation of total misses which is easily computed. All the formulae used are deduced in Section C.

It will be seen that the data relating to all the missions do not agree with expectation. The discrepancies are mutually consistent and indicate that the values of the standard errors of aiming used to obtain the expected numbers were grossly underestimating the actual level of precision. On the other hand, the agreement relating to data cleaned of gross errors seems to be perfectly satisfactory. Even the lower bounds of the frequency of total misses, though consistently lower than the figures observed, nevertheless indicate roughly the order of magnitude of this quantity.

Having in view the above results, the authors are inclined to believe that, in spite of all the difficulties encountered in measuring the bomb patterns, the results obtained by the method developed by the Analysts of the VIII Bomber Command are

accurate and could serve for predictions. Also the above results indicate that the theory of Dr. Robertson's hypothesis, if applied to suitable data, may be expected to yield results verifiable by observation.

C. Certain formulae used in the preceding section.

1. Number of hits within a rectangle. Let 2a x 2b be the dimensions of a rectangle K(x,y) centered at (x,y) with its sides parallel to the axes of coordinates. Let U_F represent the number of hits within this rectangle scored by F formations, each releasing the same number m of bombs, with the same pattern 2A x 2B, and aiming at the origin of coordinates. The purpose of the computations that follow is to deduce the first two moments of U_F for any values of a and b, A and B.

Denote by u_{ij} a random variable* defined as follows. If the jth bomb of the i-th formation hits the rectangle K(x,y), then $u_{ij} = 1$. Otherwise $u_{ij} = 0$. With this definition

$$U_F = \sum_{i=1}^{F}\sum_{j=1}^{m} u_{ij} \tag{33}$$

and the moments of U_F can be computed from those of u_{ij}. In particular,

$$\mu_1(F) = E(U_F) = \sum_{i=1}^{F}\sum_{j=1}^{m} E(u_{ij}) \tag{34}$$

*The variable u_{ij} will be described as characteristic of the experiment consisting in dropping the jth bomb. This conception will be used again.

or, since the expectations $E(u_{ij})$ have the same value for all values of the indices i and j,

$$\mu_1(F) = Fm \cdot E(u_{ij}) \tag{35}$$

Further, the second central moment of U_F, or the square of the standard error of U_F, is

$$\sigma_F^2 = \sum_{i=1}^{F}\left[\sum_{j=1}^{m}\sigma_{ij}^2 + 2\sum_{j=1}^{m-1}\sum_{k=j+1}^{m}\lambda_{ijk}\right] \tag{36}$$

where σ_{ij} stands for the standard error of u_{ij} and λ_{ijk} for the product moment of u_{ij} and u_{ik}. Denote by B_{ij} the j-th bomb of the i-th formation. In accordance with the definition of u_{ij},

$$E(u_{ij}^t) = P\{B_{ij} \epsilon K(x,y)\} = p_1 \text{ (say)}, \tag{37}$$

whatever be $t \neq 0$. Similarly, as the product $u_{ij}u_{ik}$ may have the values of unity or zero, according to whether both B_{ij} and B_{ik} hit the rectangle K(x,y) or not,

$$\begin{aligned} E(u_{ij}u_{ik}) &= P\{[B_{ij} \epsilon K(x,y)][B_{ik} \epsilon K(x,y)]\} \\ &= p_2 \text{ (say)}. \end{aligned} \tag{38}$$

It follows

$$\sigma_{ij} = p_1(1-p_1) \tag{39}$$

$$\lambda_{ijk} = p_2 - p_1^2 \tag{40}$$

Using these formulae, we get

$$\mu_1(F) = Fmp_1 \tag{41}$$

$$\begin{aligned}\sigma_F^2 &= Fm\Big(p_1(1-p_1) + (m-1)(p_2-p_1^2)\Big)\\ &= Fm\Big(p_1-mp_1^2 + (m-1)p_2\Big)\end{aligned} \tag{42}$$

The problem of computing the consecutive moments of U_F reduces to that of calculating the probabilities p_1 and p_2 of one or two particular bombs released by the same formation hitting the rectangle $K(x,y)$. To calculate these probabilities we shall need the elementary probability law of the coordinates of the points of impact of the bombs concerned.

Let ξ_j, η_j be the coordinates of the point of impact of the j-th bomb B_{ij} with respect to the axes passing through the aiming point. If X and Y denote the errors of aiming of the center C of the pattern of bombs and t_j and τ_j stand for the coordinates of B_{ij} with respect to the axes passing through the center C, then

$$\left.\begin{aligned}\xi_j &= X + t_j\\ \eta_j &= Y + \tau_j\end{aligned}\right\} \tag{43}$$

The probability law of X and Y is, say,

$$p(X,Y) = \frac{1}{\sigma_{a_d}\sqrt{2\pi}}\exp\left\{-\frac{X^2}{2\sigma_{a_d}^2}\right\}\frac{1}{\sigma_{a_r}\sqrt{2\pi}}\exp\left\{-\frac{Y^2}{2\sigma_{a_r}^2}\right\}$$

$$= \frac{1}{\sigma_{a_d}\sigma_{a_r}}g\left(\frac{X}{\sigma_{a_d}}\right)g\left(\frac{Y}{\sigma_{a_r}}\right) \tag{44}$$

The relative probability law, given X and Y, of t_1, t_2, and τ_1, τ_2 is

$$p(t_1, t_2, \tau_1, \tau_2) = \frac{1}{(4AB)^2} \text{ for } |t_j| < A, \ |\tau_j| < B$$

$$= 0 \text{ elsewhere.} \qquad (45)$$

The joint probability law of all the six variables X, Y, t_1, t_2, τ_1, τ_2 is represented by the product of (44) and (45). Introducing the variables ξ_j and η_j of (43) instead of t_j and τ_j we obtain

$$p(X,Y,\xi_1,\xi_2,\eta_1,\eta_2) = \frac{1}{\sigma_{a_d}\sigma_{a_r}}\frac{1}{(4AB)^2} g\left(\frac{X}{\sigma_{a_d}}\right)g\left(\frac{Y}{\sigma_{a_r}}\right) \text{ for } |\xi_j - X| < A, \ |\eta_j - Y| < B$$

$$= 0 \text{ elsewhere.} \qquad (46)$$

To obtain the requisite joint probability law of ξ_1, ξ_2, η_1, η_2 it is sufficient to integrate (46) for X and Y from $-\infty$ to $+\infty$. Taking into account that outside of the limits $|\xi_j - X| < A$ and $|\eta_j - B| < B$ the probability law (46) is zero, we obtain, for $\xi_j \leqq \xi_k \leqq \xi_j + 2A$ and $\eta_s \leqq \eta_t \leqq \eta_s + 2B$,

$$p(\xi_1,\xi_2,\eta_1,\eta_2) = \frac{1}{(2A)^2}\left\{G\left(\frac{\xi_j+A}{\sigma_{a_d}}\right) - G\left(\frac{\xi_k-A}{\sigma_{a_d}}\right)\right\}\frac{1}{(2B)^2}\left\{G\left(\frac{\eta_s+B}{\sigma_{a_r}}\right) - G\left(\frac{\eta_t-B}{\sigma_{a_r}}\right)\right\}$$

$$= p(\xi_1,\xi_2)p(\eta_1,\eta_2) \qquad (47)$$

and for all other systems of values of the four variables,

$$p(\xi_1, \xi_2, \eta_1, \eta_2) = 0 \qquad (47a)$$

It will be seen that, while ξ_1 and ξ_2 (also η_1 and η_2) are mutually dependent, the pair of variables ξ_1, ξ_2 is independent of the pair η_1, η_2. The joint probability law of ξ_1 and η_1 will be obtained by integrating (47) for ξ_2 and η_2, and we have

$$p(\xi_1,\eta_1) = p(\xi_1)p(\eta_1) \tag{48}$$

with

$$p(\xi_1) = \frac{1}{2A}\{G(\frac{\xi_1+A}{\sigma_{a_d}}) - G(\frac{\xi_1-A}{\sigma_{a_d}})\} \tag{49}$$

and a similar formula for $p(\eta_1)$.

To obtain the probability p_1 that a specified bomb will hit the rectangle $K(x,y)$ it is sufficient to integrate $p(\xi_1)$ between the limits $x-a$ and $x+a$ and to multiply the result by a similar integral of $p(\eta_1)$ taken from $y-b$ to $y+b$.

Similarly, the probability p_2 of two specified bombs falling within the rectangle $K(x,y)$ is obtained as a product of the integrals of $p(\xi_1,\xi_2)$ and $p(\eta_1,\eta_2)$. The first integral is taken over the common part of the square $|\xi_j-x| \leqq a$ and of the region $\xi_j \leqq \xi_k \leqq \xi_j+2A$; and the second over the common part of the square $|\eta_j-y| < b$ and of the region $\eta_j \leqq \eta_k \leqq \eta_j + 2B$, with $j,k = 1,2$. Using the formula

$$\int_\alpha^\beta G(t-\gamma)dt = \Big[(t-\gamma)G(t-\gamma) + g(t-\gamma)\Big]_\alpha^\beta = \Pi(t-\gamma)\Big]_\alpha^\beta \text{ (say)} \tag{50}$$

the integration is easily performed. In particular

$$p_1 = P\{B_{ij} \epsilon K(x,y)\} = P\{x-a < \xi < x+a\} \cdot P\{y-b < \eta < y+b\}$$

$$= \frac{\sigma_{a_d}}{2A}\{ \Pi(\frac{x+a+A}{\sigma_{a_d}}) - \Pi(\frac{x-a+A}{\sigma_{a_d}}) - \Pi(\frac{x+a-A}{\sigma_{a_d}}) + \Pi(\frac{x-a-A}{\sigma_{a_d}})\} \cdot$$

$$\cdot \frac{\sigma_{a_r}}{2B}\{ \Pi(\frac{y+b+B}{\sigma_{a_r}}) - \Pi(\frac{y-b+B}{\sigma_{a_r}}) - \Pi(\frac{y+b-B}{\sigma_{a_r}}) + \Pi(\frac{y-b-B}{\sigma_{a_r}})\} \qquad (51)$$

For convenience of computations Figure 7 was constructed, giving the values of $\Pi(\pm x)$ for values of $|x| < 2.2$. For greater values of $|x|$ the value of $\Pi(\pm x)$ is indistinguishable from $\frac{1}{2}|x|$.

The formula for p_2 is a little more complex. As was the case with p_1, the probability p_2 is a product of two values of the same function

$$p_2 = f(x,a,A,\sigma_{a_d})f(y,b,B,\sigma_{a_r}) \qquad (52)$$

but the function f has a form which differs according to whether the second of its arguments is less or greater than the third. Thus, if $a \leqq A$, then

$$f(x,a,A,\sigma_{a_d}) = \frac{\sigma^2_{a_d}}{4A^2}\{\frac{(x+a+A)}{\sigma_{a_d}} \Pi(\frac{x+a+A}{\sigma_{a_d}}) + G(\frac{x+a+A}{\sigma_{a_d}})$$

$$+ \frac{(x-3a-A)}{\sigma_{a_d}} \Pi(\frac{x+a-A}{\sigma_{a_d}}) + G(\frac{x+a-A}{\sigma_{a_d}})$$

$$- \frac{(x+3a+A)}{\sigma_{a_d}} \Pi(\frac{x-a+A}{\sigma_{a_d}}) - G(\frac{x-a+A}{\sigma_{a_d}})$$

$$- \frac{(x-a-A)}{\sigma_{a_d}} \Pi(\frac{x-a-A}{\sigma_{a_d}}) - G(\frac{x-a-A}{\sigma_{a_d}})\} \qquad (53)$$

On the other hand, if $A \leqq a$, then

$$\Pi(x) = x\,G(x) + g(x) = \frac{x}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}\,dt + \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}$$

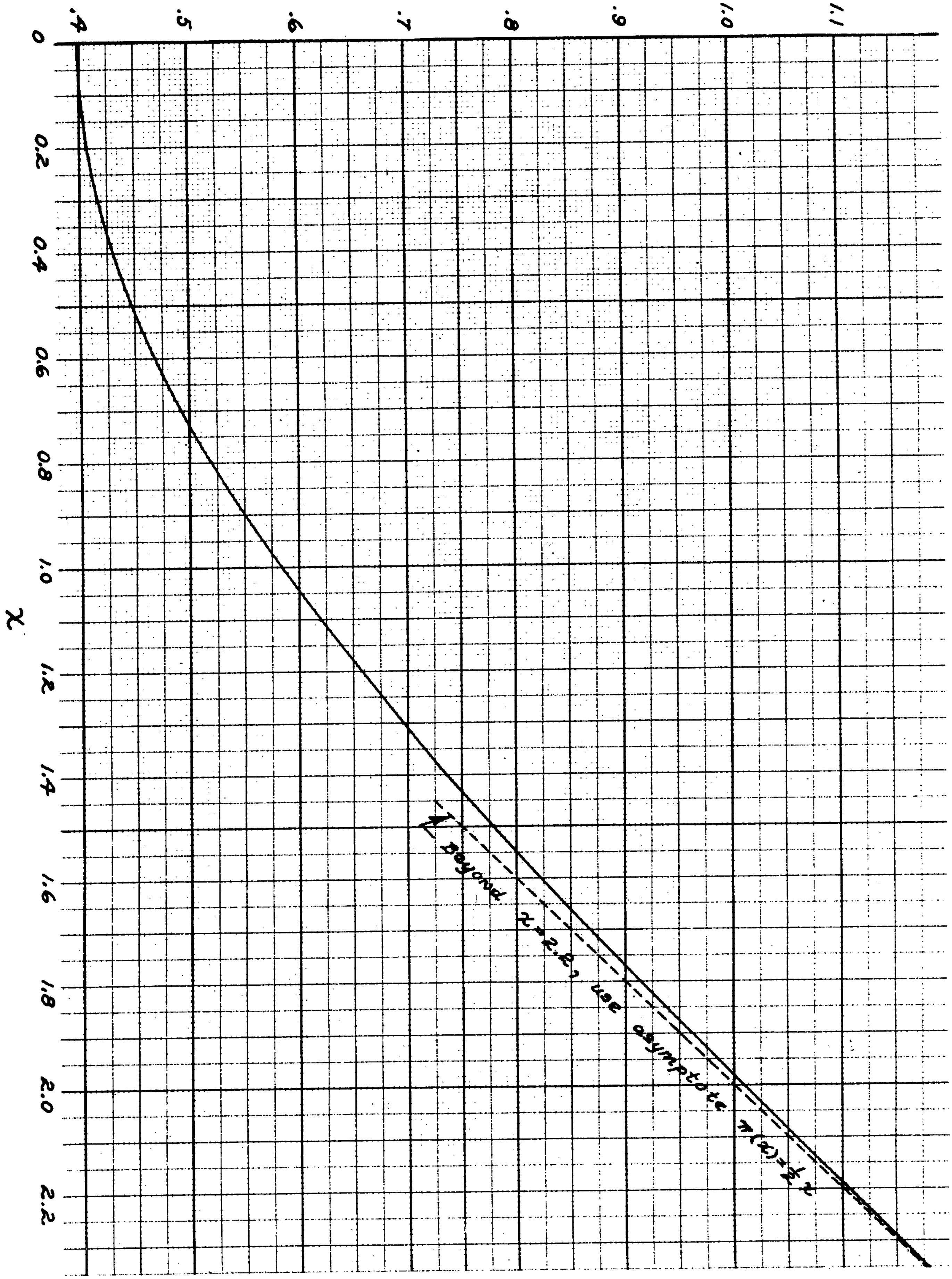

Figure 7

$$f(x,a,A,\sigma_{a_d}) = \frac{\sigma_{a_d}^2}{4A^2}\Big\{\frac{(x+a+A)}{\sigma_{a_d}}\Pi\Big(\frac{x+a+A}{\sigma_{a_d}}\Big) + G\Big(\frac{x+a+A}{\sigma_{a_d}}\Big)$$

$$+ \frac{(x-a-3A)}{\sigma_{a_d}}\Pi\Big(\frac{x-a+A}{\sigma_{a_d}}\Big) + G\Big(\frac{x-a+A}{\sigma_{a_d}}\Big)$$

$$- \frac{(x+a+3A)}{\sigma_{a_d}}\Pi\Big(\frac{x+a-A}{\sigma_{a_d}}\Big) - G\Big(\frac{x+a-A}{\sigma_{a_d}}\Big)$$

$$- \frac{(x-a-A)}{\sigma_{a_d}}\Pi\Big(\frac{x-a-A}{\sigma_{a_d}}\Big) - G\Big(\frac{x-a-A}{\sigma_{a_d}}\Big)\Big\} \qquad (54)$$

The two formulae are a little long. However, with practice and the help of a slide rule and of Figures 7 and 8 (Figure 8 gives the values of $G(x)$), the computations are easy. It is just possible that a convenient nomogram could be constructed giving the values of $E(U_F)$ and of σ_{U_F} at once.

2. Probability of a total miss. Of the formulae used in Section B there remains to be deduced only one, giving the lower bound of the probability that all the m bombs released by a formation will miss the rectangle $K(x,y)$. The lower bound used is represented by the probability

$$1 - P\{x-a-A < X < x+a+A\}\cdot P\{y-b-B < Y < y+b+B\} \qquad (55)$$

that the center of the bomb pattern τ falls so far from the point (x,y) that the pattern τ has no points in common with the rectangle $K(x,y)$. In this case, the computations do not present any difficulty since, for example,

$$P\{x-a-A < X < x+a+A\} = G\Big(\frac{x+a+A}{\sigma_{a_d}}\Big) - G\Big(\frac{x-a-A}{\sigma_{a_d}}\Big) \qquad (56)$$

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$$G(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}}\,dt$$

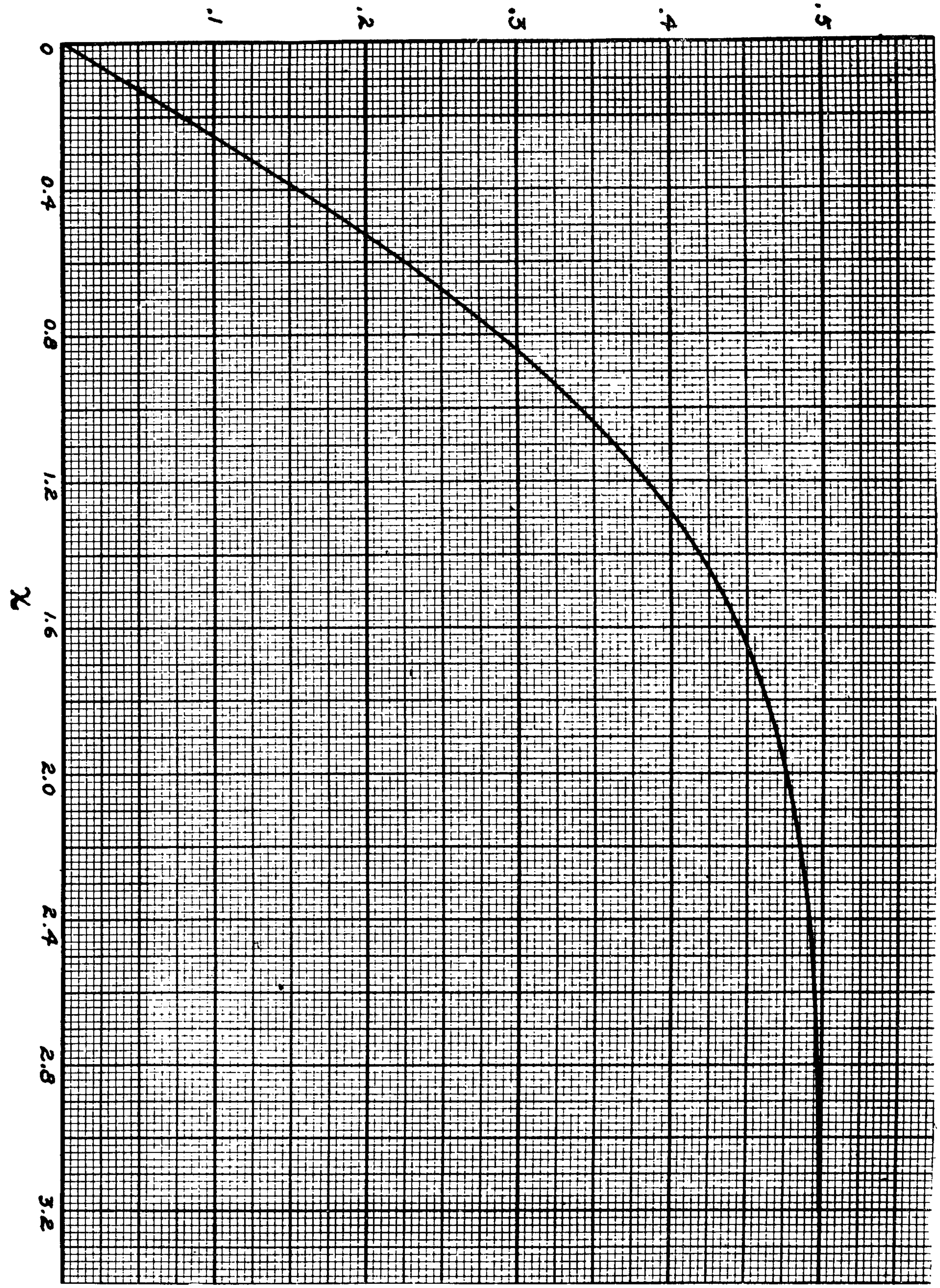

Figure 8

and the formula for the coordinate Y is similar. It is obvious that the greater the value of m, the closer the value of the computed lower bound will be to the actual probability of missing the rectangle K(x,y).

It will be noticed that, if the rectangle K(x,y) is centered at the point of aiming so that $x = y = 0$, then all of the above formulae simplify somewhat. However, there is an advantage in having the formulae for arbitrary x and y because, in cases of complex targets, it is frequently interesting to obtain an idea of the chances of hits on some particular sub-target.

In using the above formulae it is convenient to remember that G(x) is an odd function so that

$$G(-x) = -G(x). \tag{57}$$

On the other hand both g(x) and $\Pi(x)$ are even functions, and

$$g(x) = g(-x) > 0 \tag{58}$$

$$\Pi(x) = \Pi(-x) \geq \Pi(0) \tag{59}$$

D. An attempt to solve question (b).

1. General. In this subsection an attempt is made to answer the question: What values of pattern dimensions 2A x 2B (and possibly also of other parameters) should be used so that the theory of the Robertson hypothesis will yield results referring to specified conditions of bombing such as the size and shape of the formation, the number and the spacing of bombs in train, etc.?

The method used in this attempt is based on the more

detailed idealization of the formation bombing, which was used in the First Cooperative Study. As this more detailed theory ignores fewer of the relevant indisputable facts pertaining to formation bombing than the hypothesis of Robertson, the probabilities implied by the former must agree with the actual relative frequencies better than those implied by the latter. Assuming this, the problem of determining A and B so that the Robertson's probabilities agree with the actual frequencies is reduced to that of determining A and B so that the Robertson's probabilities are approximately equal to the probabilities implied by the more detailed theory. While this seems to be the best that can be done, the authors are aware that the more detailed theory of formation bombing is itself an idealization and that its consequences need not be in agreement with the actual frequencies.

The hypotheses underlying the more detailed theory of formation bombing are as follows.

Denote by (OX,OY) the rectangular system of axes of coordinates with their origin at the point O, the intended center of gravity of all the bombs dropped. The point O will be described as the aiming point. The direction of the axis OY is that of the line of flight. The axes (OX,OY) will be described as fixed. Apart from the fixed system of axes another system $(C\xi, C\eta)$ will be considered, the origin of which, C, will be described as the center of the bomb pattern. To define these axes assume for a

moment that the aiming is without error and that the pattern of actual points of impact of all bombs coincides with the intended one. When C coincides with O, the axis $C\xi$ coincides with OX and the axis $C\eta$ with OY. Denote by ξ_i , η_{ij} the coordinates of the intended point of impact of the j-th bomb to be released by the i-th plane participating in the formation, for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, n$. Let

$$\eta_{i\cdot} = \frac{1}{n} \sum_{j=1}^{n} \eta_{ij} \tag{60}$$

It is obvious that ξ_i and $\eta_{i\cdot}$ represent the coordinates of the intended center of gravity of the train to be released by the i-th plane. The coordinate ξ_i depends only on the intended position of the i-th plane within the formation. On the other hand the coordinate $\eta_{i\cdot}$ depends both on the intended position of the i-th plane and on the intended time interval between the moments of release of the leader and of the i-th plane. The differences $\eta_{i,j+1} - \eta_{i,j} = 2u$ are equal to the spacing of bombs in the trains and are assumed to be uniform.

The coordinates x_{ij}, y_{ij} of the actual point of impact of the j-th bomb released by the i-th plane, taken with respect to the fixed axes, are considered to be of the form

$$\begin{aligned} x_{ij} &= X + \xi_i + f_i + \delta_{ij} \\ y_{ij} &= Y + \eta_{ij} + F_i + \epsilon_{ij} \end{aligned} \tag{61}$$

Here the letters on the right hand side have the following meaning. X and Y, described as errors of aiming for deflection and for range respectively, denote the coordinates of the actual center C of the bomb pattern. The letters f_i and F_i , described as formation errors of the i-th plane, denote the displacements of the actual center of the i-th train from its intended position with respect to the axes $C\xi$ and $C\eta$. Finally, δ_{ij} and ϵ_{ij} represent the dispersion errors of the j-th bomb released by the i-th plane.

The equations (61) do not imply any hypothesis. The basic hypothesis underlying the more detailed theory of formation bombing is that the errors of aiming X and Y, the formation errors f_i and F_i, and the dispersion errors δ_{ij} and ϵ_{ij} are all mutually independent random variables, normally distributed about zero with S.E.'s respectively σ_{a_d}, σ_{a_r}, σ_f, σ_F, σ_{d_d} and σ_{d_r}. This hypothesis implies that

$$E(x_{ij}) = \xi_i, \quad E(y_{ij}) = \eta_{ij} \tag{62}$$

$$\sigma_{x_{ij}} = \sqrt{\sigma^2_{a_d} + \sigma^2_f + \sigma^2_{d_d}} = \sigma_x \text{ (say)} \tag{63}$$

$$\sigma_{y_{ij}} = \sqrt{\sigma^2_{a_r} + \sigma^2_F + \sigma^2_{d_r}} = \sigma_y \text{ (say)} \tag{64}$$

Also it follows from the basic hypothesis that every variable x_{ij} is independent of every variable $y_{t\tau}$. On the other hand, x_{ij} is correlated with $x_{t\tau}$ and y_{ij} with $y_{t\tau}$.

2. Expected proportion of bombs hitting a given target. Let P(S) stand for the expected proportion of bombs, released by one formation, falling within any given area S. The method of variables* characteristic to the experiments consisting in releasing particular bombs gives then immediately,

$$P(S) = \frac{1}{Nn} \sum_{i=1}^{N} \sum_{j=1}^{n} \iint_{S} p_{ij}(t,\tau)\,dt\,d\tau \tag{65}$$

where

$$p_{ij}(t,\tau) = \frac{1}{\sigma_x} g\left(\frac{t-\xi_i}{\sigma_x}\right) \frac{1}{\sigma_y} g\left(\frac{\tau-\eta_{ij}}{\sigma_y}\right) \tag{66}$$

stands for the elementary probability law of x_{ij} and y_{ij}. The general formula (65) simplifies if both the target area S and the intended pattern of bombs have certain characteristics of regularity. If S is a rectangle, say K(x,y), centered at a point (x,y) with its sides 2a x 2b parallel to the axes of coordinates, then

$$\int\int_{K(x,y)} p_{ij}(t,\tau)\,dt\,d\tau = \left(G\left(\frac{x+a-\xi_i}{\sigma_x}\right) - G\left(\frac{x-a-\xi_i}{\sigma_x}\right)\right)\left(G\left(\frac{y+b-\eta_{ij}}{\sigma_y}\right) - G\left(\frac{y-b-\eta_{ij}}{\sigma_y}\right)\right) \tag{67}$$

If the intended bomb pattern is approximated by a rectangular lattice of equidistant points (ξ_ν, η_μ), $\nu = 1, 2, \ldots, s$; $\mu = 1, 2, \ldots, n$, where s stands for the number of columns or

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*See footnote on p. 43.

strings of planes in the formation, with the lateral spacing between the strings equal to 2d, then $P\left(S=K(x,y)\right)$ becomes the product of two sums

$$P\left(K(x,y)\right) = \frac{1}{s}\sum_{\nu=1}^{s}\{G(\frac{x+a-\xi_\nu}{\sigma_x})-G(\frac{x-a-\xi_\nu}{\sigma_x})\}\cdot\frac{1}{n}\sum_{\mu=1}^{n}\{G(\frac{y+b-\eta_\mu}{\sigma_y})-G(\frac{y-b-\eta_\mu}{\sigma_y})\} \tag{68}$$

Ordinarily both the lateral spacing between planes and the spacing of bombs in the trains are smaller than σ_x and σ_y respectively. In these conditions it is known that the two sums in (68) can be replaced by the integrals divided by 2d and 2u respectively, with the result that P(K) computed from (68) differs only by a unit or two in the third decimal from the value P'(K) computed from(69):

$$P'(K) = \frac{1}{2sd}\int_{-sd}^{+sd}\{G(\frac{x+a-\xi}{\sigma_x})-G(\frac{x-a-\xi}{\sigma_x})\}d\xi\cdot\frac{1}{2nu}\int_{-nu}^{+nu}\{G(\frac{y+b-\eta}{\sigma_y})-G(\frac{y-b-\eta}{\sigma_y})\}d\eta \tag{69}$$

Performing an easy integration it is found that P'(K) is identical with the value p_1 obtained in Subsection C,1 on Robertson's hypothesis, provided one substitutes

$$A = sd,\ B = nu,\quad \sigma_{a_d} = \sigma_x,\quad \sigma_{a_r} = \sigma_y \tag{70}$$

The Robertson formula applies to C-bombs only. On the other hand, formula (69) refers to all the bombs dropped. Thus the identity is achieved by assuming that all the bombs dropped are C-bombs

and thus that $\alpha = 1$ and $m = nN$. If the intended pattern of bombs is represented by a rectangular lattice with the spacings 2u in range and 2d in deflection*, then the Robertson formula for the expected proportion of bombs falling within a rectangle K(x,y) will yield values approximately equal to those given by the more detailed theory, provided the dimensions of the Robertson rectangular pattern of bombs are set equal to

$$2A = 2sd, \qquad 2B = 2nu \tag{71}$$

and provided the standard errors of aiming for deflection and for range in the Robertson's formula are identified with the somewhat larger quantities

$$\sigma_x = \sqrt{\sigma_{a_d}^2 + \sigma_f^2 + \sigma_{d_d}^2} \quad \text{and} \quad \sigma_y = \sqrt{\sigma_{a_r}^2 + \sigma_F^2 + \sigma_{d_r}^2} \tag{72}$$

Frequently the standard errors of dispersion and occasionally those of formation errors will be small compared to the standard errors of aiming. In such cases the ratios σ_x/σ_{a_d} and σ_y/σ_{a_r} will be close to unity. This however, will not always be the case.

*It will be noticed that this does not imply that the formation pattern necessarily must be rectangular. On the contrary, if all the planes release their bombs on the leader, then, owing to the delay in the time of release, to obtain a rectangular pattern of bombs, it is essential that the wing planes be staggered in range from the leading plane. By adjusting the lag in time of release a rectangular intended pattern of bombs may easily be achieved by a formation composed of several "Vee's".

In fact, three bomb plots obtained in experimental bombing gave the following estimates of formation errors:

$$\sigma_f = 100'$$

$$\sigma_F = 500'.$$

In combat, even larger values of the S.E.'s than these may occur and then σ_x and/or σ_y will be appreciably greater than σ_{a_d} and σ_{a_r} respectively. It should be mentioned that, while the dispersion S.E.'s of G.P. bombs appear to be small compared with the S.E.'s of aiming, the dispersion of the 20 lb. fragmentation bombs (on which the authors do not have any experimental data) is likely to be very large, of the order of 100' or more.

Whatever the situation may be, the problem of the arguments to be used in the Robertson theory to obtain the expected proportion of bombs hitting a given rectangle, which is implied by the more detailed theory, seems to be solved by the rule (71) and (72) explained above.

3. <u>Probability of missing a circle S(x,y)</u>. The probability $\mathfrak{P}_{1,0}$ that all the bombs released by one formation will miss a circle S(x,y), implied by the more detailed theory, is given by the formula

$$\mathfrak{P}_{1,0}(x,y) = \frac{1}{\sigma_{a_d}\sigma_{a_r}} \int_{-\infty}^{+\infty}\!\!\int \Bigg\{ g\left(\frac{X}{\sigma_{a_d}}\right) g\left(\frac{Y}{\sigma_{a_r}}\right) \prod_{i=1}^{N} \frac{1}{\sigma_f\sigma_F} \int_{-\infty}^{+\infty}\!\!\int g\left(\frac{u-X-\xi_i}{\sigma_f}\right) \cdot g\left(\frac{v-Y-\eta_i}{\sigma_F}\right) \prod_{j=1}^{n} (1-\Pi_{ij})\,du\,dv \Bigg\} dX\,dY \qquad (73)$$

where ξ_i and η_i denote the coordinates of the intended center of the train of bombs released by the i-th plane, and

$$\Pi_{ij} = \frac{1}{\sigma_{d_d}\sigma_{d_r}} \int\int_{S(x,y)} g\left(\frac{t-u+X+\xi_i}{\sigma_{d_d}}\right) \cdot g\left(\frac{\tau-v+Y+\eta_{ij}}{\sigma_{d_r}}\right) dt\, d\tau. \tag{74}$$

Obviously, the probability of missing by F formations is given by $\mathfrak{P}_{F,0} = \mathfrak{P}^F_{1,0}$.

The formula (73) is complex and so far it has been impossible to establish a relation to the corresponding formula of the Robertson theory that would guarantee the approximate agreement of the two, provided A and B in Robertson's theory are given some values calculable from the parameters involved in (73). Although work in the above direction is in progress at the moment of writing, the best that can be reported are the results of a few trials. The first trial, suggested by the results obtained in the previous subsection D-2, was based on the rule

$$\alpha=1, \quad A = sd, \quad B = nu, \quad \sigma_{a_d} = \sigma_x, \quad \sigma_{a_r} = \sigma_y, \tag{75}$$

which brought about the approximate identity of the expected number of bombs hitting a rectangle, computed on the two theories. While the totality of numerical examples comparing $\mathfrak{P}_{F,0}(x,y)$ and $P^*_{F,0}(x,y)$ is given later in a comprehensive table, Table XI gives four examples indicating that $P^*_{F,0}$ computed with the sub-

stitution (75) may have values very different from those implied by the formula (73). In all four examples the number F of formations was chosen to obtain values of $\mathfrak{P}_{F,0}$ ranging from about .05 to .15.

The first two examples use values of the parameters assumed tentatively to be appropriate for 500 lb. G. P. bombs, perhaps with slightly exaggerated dispersion. This, however, need not be unreal if the bombing is done from high altitudes. The two other examples refer to the 20 lb. fragmentation bomb for which the maximum load is n = 144 per plane. Here the assumed value of dispersion (the same as in the first two examples) may be a little too small*. This choice was made in order to shorten the computations, the main purpose of which is to test the agreement between $\mathfrak{P}_{F,0}$ and $P^*_{F,0}$.

In all computations, the lateral spacing of planes is 110' in accordance with the usual practice described in Prelimary Report No. 9 of the Operations Analysis Section, Thirteenth Air

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*Since these computations were performed, and after this passage was written, the authors obtained a copy of the Report dated March 31st, 1944 of the Operations Analysis Section of the Fifteenth Air Force, which includes plots of points of impact of 20 lb. fragmentation bombs. These plots indicate that the dispersion S.E.'s of these bombs are of the order $\sigma_{d_d} = \sigma_{d_r} = 100'$. It follows that the assumed values overestimate the dispersion of both the 500 lb. and the 20 lb. bombs. Since the change in σ_{d_d} and σ_{d_r} from 50' to 150' produces but a very moderate change in the values of σ_x and σ_y, namely from 808' to 820' and from 945' to 955' respectively, it is doubtful whether the overestimate of the dispersion standard errors is of any importance.

TABLE XI

Comparison of $\mathfrak{P}_{F,0}(x,y)$ and $P^*_{F,0}(x,y)$ Computed with Parameters which Identify the Values of p_1 Implied by the Two Theories.

General assumptions: $\sigma_{a_d} = \sigma_{a_r} = 800'$, $\sigma_f = 100'$, $\sigma_F = 500'$

$\sigma_{d_d} = \sigma_{d_r} = 150'$, $N = 6$; $x = 20'$.

Example:	1		2		3		4	
	s=3, 2d=110' n=12, 2u=240' R = 60' F = 20		s=3, 2d=110' n=16, 2u=180' R = 60' F = 20		s=3, 2d=110' n=144, 2u=20' R = 30' F = 10		s=6, 2d=110' n=144, 2u=20' R = 30' F = 10	
y	$\mathfrak{P}_{F,0}$	$P^*_{F,0}$	$\mathfrak{P}_{F,0}$	$P^*_{F,0}$	$\mathfrak{P}_{F,0}$	$P^*_{F,0}$	$\mathfrak{P}_{F,0}$	$P^*_{F,0}$
200'	.107	.193	.056	.141	.078	.258	.042	.115
300'	.111	.197	.058	.144	.080	.263	.044	.119
400'	.115	.203	.061	.150	.084	.270	.046	.124
500'	.120	.211	.065	.156	.091	.278	.050	.130
600'	.128	.220	.070	.165	.093	.288	.054	.138

Force.

It will be seen that the values of $P^*_{F,0}$ are twice or even three times as large as those of $\mathcal{P}_{F,0}$.* It follows that substitution (75) is not satisfactory from the point of view of agreement between the values of $P^*_{F,0}$ and $\mathcal{P}_{F,0}$ and that, therefore, attempts to find a better substitution are justified. In addition to the method just described, which may be labeled method (a), the following three methods were tried.

(b) Use $\alpha = 1$, $A = sd$, $B = nu$, as in method (a), but substitute σ_{a_d} and σ_{a_r} instead of σ_x and σ_y as previously used.

(c) Use $\alpha = 1$,

$$A = \sqrt{(s^2-1)d^2 + 3(\sigma_f^2 + \sigma_{d_d}^2)} \qquad (76)$$

$$B = \sqrt{(n^2-1)u^2 + 3(\sigma_F^2 + \sigma_{d_r}^2)}$$

and σ_{a_d}, σ_{a_r}. This implies that A and B are selected so that the second moment of the x coordinate (and y) of the points of impact of bombs about the center of gravity of the pattern computed on one theory is equal to the corresponding moment computed on the

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*Although the above computations were made to compare $\mathcal{P}_{F,0}$ with $P^*_{F,0}$, rather than to draw some tactical conclusions, it is interesting to note that examples 3 and 4 suggest that one "Vee" of six planes releasing fragmentation bombs is much more effective than a formation of two "Vee's" of three planes each, following each other.

other theory. Similar substitutions were tried, based on the equality of moments of other orders. However, while the results were not markedly better, the formulae for computation of A and B were considerably more complex.

(d) Use $\alpha = 1$,

$$A = (s-1)d + \tfrac{1}{2}w_n \sqrt{\sigma_f^2 + \sigma_{a_d}^2},$$
$$B = (n-1)u + \tfrac{1}{2}w_N \sqrt{\sigma_F^2 + \sigma_{a_r}^2}, \tag{77}$$

and σ_{a_d}, σ_{a_r}; where w_m means the expected range in a sample of m items independently drawn from a normal population with unit standard deviation. This implies that A and B are selected so that 2A (and 2B) equal the expected difference between the greatest and smallest values of the x coordinate (and y) of all bombs. An extensive table of values of w_n is given in Table XXII in the Tables for Statisticians and Biometricians, Part II, by Karl Pearson.

Table XII gives the results obtained. It will be seen that most of them refer to small values of R, σ_f and σ_F, the latter two being out of proportion to the estimates $\sigma_f = 100'$ and $\sigma_F = 500'$ computed from experimental data. The small values used in Table XII, $\sigma_f = 10'$ and $\sigma_F = 25'$, are estimates made before any experimental data were available and which were used in practically all

TABLE XII

Comparison of Values of $P^*_{F,0}$, computed by Four Methods, with Those of $\mathcal{P}_{F,0}$.

PART I: General Conditions: $m = 864$, $\sigma_a = 800'$, $\sigma_f = 100'$
$\sigma_F = 500'$, $x = 20'$, $F = 10$, $\sigma_{d_d} = \sigma_{d_r} = 150'$, $2u = 20'$

R	s	2d	y	$\mathcal{P}_{F,0}$	$P^*_{F,0}$ computed using method (a)	(b)	(c)	(d)
30	3	110	200	.078	.258	.225	.097	.097
			300	.080	.263	.229	.099	.098
			400	.084	.270	.236	.101	.099
			500	.091	.278	.244	.105	.100
			600	.093	.288	.254	.110	.103
	3	360	200	.059	.074	.056	.067	.101
			300	.062	.077	.058	.069	.102
			400	.064	.081	.062	.071	.103
			500	.069	.086	.066	.074	.104
			600	.074	.093	.073	.078	.106
	6	110	200	.042	.115	.091	.078	.056
			300	.044	.119	.094	.080	.056
			400	.046	.124	.099	.082	.057
			500	.050	.130	.105	.085	.058
			600	.054	.138	.113	.090	.060

TABLE XII - continued.

Comparison of Values of $P^{*}_{F,0}$, computed by Four Methods, with Those of $\mathcal{P}_{F,0}$.

PART II: General Conditions: $s = 3$, $\sigma_a = 800'$, $\sigma_f = 100'$. $\sigma_F = 500'$, $x = 20'$, $F = 20$, $\sigma_{d_d} = \sigma_{d_r} = 150'$, $2d = 110'$.

R	m	2u	y	$\mathcal{P}_{F,0}$	$P^{*}_{F,0}$ computed using method (a)	(b)	(c)	(d)
60	72	240	200	.107	.193	.164	.138	.131
			300	.111	.197	.168	.140	.132
			400	.115	.203	.173	.143	.134
			500	.120	.211	.181	.148	.136
			600	.128	.220	.190	.153	.139
	96	180	200	.056	.141	.116	.079	.101
			300	.058	.144	.119	.081	.101
			400	.061	.150	.124	.083	.103
			500	.065	.156	.130	.086	.104
			600	.070	.165	.138	.090	.107

PART III: General Conditions: $s = 3$, $\sigma_a = 400'$, $\sigma_f = 100'$, $\sigma_F = 500'$, $x = 20'$, $F = 30$, $\sigma_{d_d} = 30'$, $\sigma_{d_r} = 50'$, $2d = 100'$

R	m	2u	y	$\mathcal{P}_{F,0}$	$P^{*}_{F,0}$ computed using method (a)	(b)	(c)	(d)
10	360	100	200	.263	.276	.222	.309	.440
			400	.285	.300	.237	.312	.440
			600	.323	.340	.274	.321	.440
16	216	100	200	.133	.121	.056	.114	.202
			300	.139	.134	.069	.118	.202
			400	.149	.152	.092	.124	.203
			500	.163	.177	.126	.133	.204

TABLE XII - continued.

Comparison of Values of $P^{*}_{F,O}$, computed by Four Methods, with Those of $\mathfrak{P}_{F,O}$.

PART IV: General Conditions: $s = 9$, $\sigma_a = 400'$, $\sigma_f = 50'$, $\sigma_F = 100'$, $x = 20'$, $\sigma_{d_d} = 30'$, $\sigma_{d_r} = 50'$, $2d = 100'$

R	m	F	y	2u	$\mathfrak{P}_{F,O}$	$P^{*}_{F,O}$ computed using method (a)	(b)	(c)	(d)
10	180	60	200	20	.087	.099	.091	.088	.095
				40	.097	.101	.094	.099	.119
				60	.124	.127	.120	.128	.157
				80	.166	.169	.163	.160	.206
				100	.217	.220	.215	.222	.260
			400	20	.172	.189	.185	.173	.173
				40	.165	.171	.165	.166	.176
				60	.174	.178	.171	.175	.195
				80	.197	.200	.193	.199	.227
				100	.233	.236	.230	.236	.269
			600	20	.359	.389	.380	.360	.342
				40	.311	.320	.322	.312	.298
				60	.276	.282	.278	.276	.274
				80	.262	.266	.260	.262	.274
				100	.270	.274	.266	.270	.293
10	360	30	200	20	.103	.132	.122	.106	.108
				40	.106	.116	.107	.110	.131
			400	20	.196	.233	.228	.200	.186
				40	.177	.190	.184	.181	.187
				60	.182	.189	.182	.185	.206
			600	60	.287	.297	.293	.288	.279
				80	.270	.276	.269	.270	.281
				100	.276	.281	.274	.277	.302

TABLE XII - continued.

Comparison of Values of $P^*_{F,0}$, Computed by Four Methods, with Those of $\mathcal{P}_{F,0}$.

PART V: General Conditions: $s = 3$, $\sigma_a = 400'$, $\sigma_f = 10'$, $\sigma_F = 25'$ $x = 20'$, $\sigma_{d_d} = 30'$, $\sigma_{d_r} = 50'$, $2d = 100'$.

R	m	F	y	2u	$\mathcal{P}_{F,0}$	$P^*_{F,0}$ computed using method (a)	(b)	(c)	(d)
10	180	60	200	20	.090	.104	.101	.097	.092
				40	.094	.098	.096	.097	.101
				60	.118	.122	.120	.121	.129
			400	40	.164	.170	.168	.168	.166
				60	.168	.172	.170	.170	.174
				80	.189	.192	.190	.191	.197
			600	80	.254	.258	.256	.256	.255
				100	.260	.264	.262	.262	.265
10	360	30	200	20	.129	.155	.152	.141	.122
				40	.121	.121	.119	.118	.120
				60	.136	.138	.136	.136	.146
			400	40	.200	.201	.199	.197	.189
				60	.189	.191	.190	.189	.191
				80	.204	.207	.205	.205	.212
			600	80	.271	.275	.273	.272	.269
				100	.272	.276	.274	.274	.278
16	180	60	200	20	.007	.014	.013	.010	.007
				40	.004	.006	.006	.003	.005
				60	.007	.007	.007	.007	.008
			400	40	.016	.020	.020	.012	.017
				60	.015	.017	.016	.016	.016
				80	.018	.020	.019	.019	.020
			600	80	.036	.040	.039	.039	.038
				100	.036	.040	.039	.039	.039

(Part V continued on p.69)

TABLE XII - concluded.

Comparison of Values of $P^{*}_{F,0}$, Computed by Four Methods, with Those of $\mathfrak{P}_{F,0}$.

PART V (continued): General Conditions: $s = 3$, $\sigma_a = 400'$, $\sigma_f = 10'$, $\sigma_F = 25'$, $x = 20'$, $\sigma_{d_d} = 30'$, $\sigma_{d_r} = 50'$, $2d = 100'$

R	m	F	y	2u	$\mathfrak{P}_{F,0}$	$P^{*}_{F,0}$ computed using method (a)	(b)	(c)	(d)
16	360	30	200	20	.033	.059	.058	.045	.024
				40	.012	.017	.017	.016	.013
				60	.019	.015	.014	.014	.015
			400	40	.035	.047	.046	.043	.034
				60	.035	.030	.030	.029	.027
				80	.032	.030	.029	.029	.029
			600	80	.058	.057	.056	.056	.051
				100	.051	.053	.052	.051	.051

the computations given in the First Cooperative Study. At present the results of these computations are used solely for purposes of comparison of $P^{*}_{F,0}$ with $\mathfrak{P}_{F,0}$.

It will be seen that the values of $P^{*}_{F,0}$ agreeing most consistently with $\mathfrak{P}_{F,0}$ in order of magnitude are those provided by method (c) which is based on equating the second moments of the distribution of bombs within the pattern, computed on the two theories. It is true that the agreement is not perfect, but it

seems probable that, whatever practical conclusions would have been made on the basis of the values of $\mathfrak{P}_{F,0}$, the same conclusions would be reached on the basis of the values of $P^{*}_{F,0}$.

The unpleasant detail of the situation is that at the present moment the authors are unable to give any assurance that in some untried circumstances the values of $P^{*}_{F,0}$ will not differ disastrously from those of $\mathfrak{P}_{F,0}$. Therefore, the method (c) is offered very tentatively with the hope that some future research will bring an improvement and/or an assurance of consistency.

Figures 9, 10 and 11 give bomb plots obtained in sampling experiments with the patterns fitted to them by the various methods. These methods are also illustrated in Figures 4, 5 and 6.

Before concluding this section the authors wish to emphasize that the problem of the correct values of arguments in the Robertson theory has received only a tentative solution without a proper theoretical backing. In their opinion it would be very important to improve the situation in this respect, because the theory of the Robertson hypothesis leads to easy formulae and nomograms which solve a great variety of practical problems.

VI. OPTIMUM BOMB PATTERN.

A. General. The Report of the VIII B.C. refers to the problem of the optimum bomb pattern several times. Two passages from this Report are quoted:

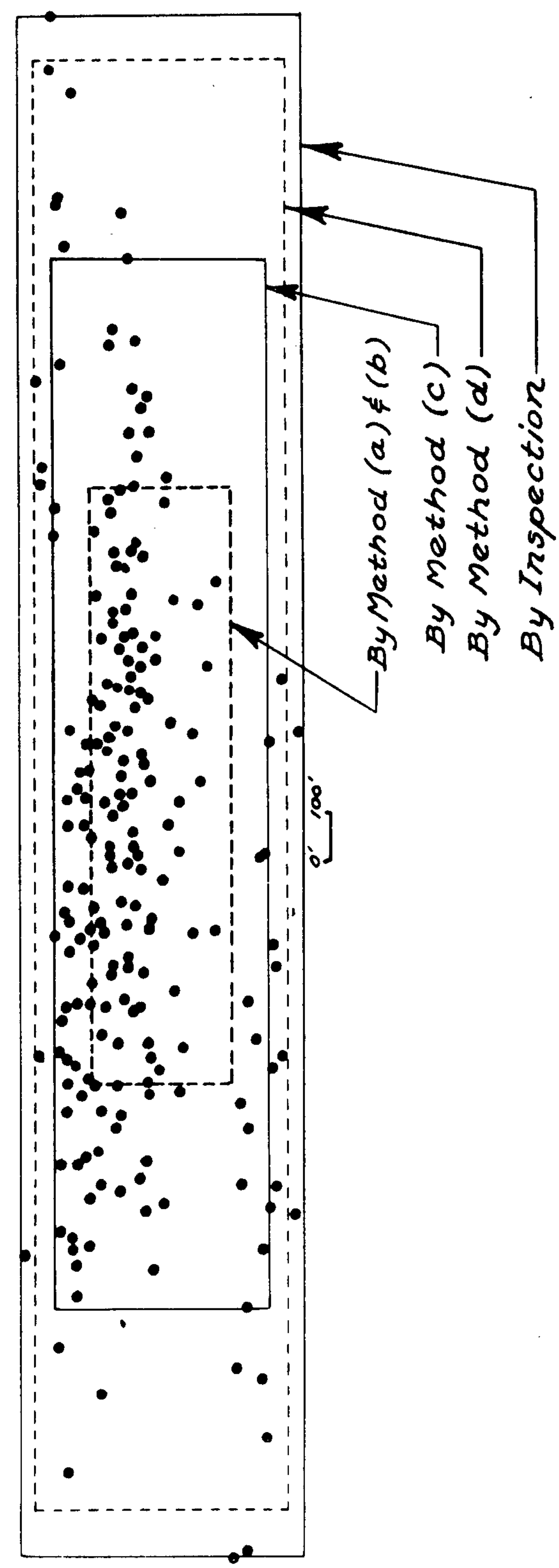

Figure 9

Bomb Plot. Sampling Experiment.

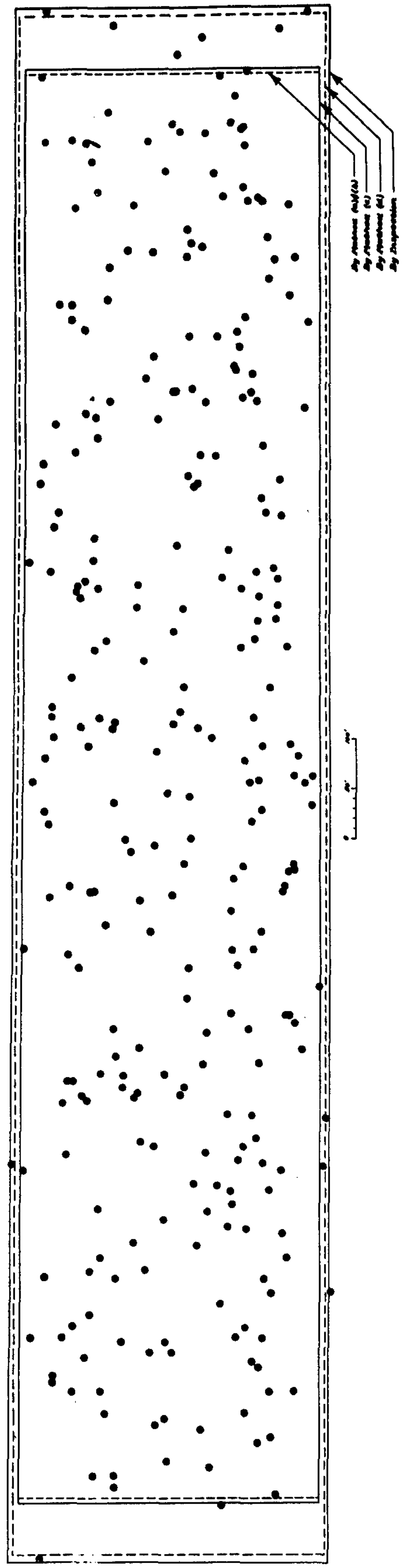

Figure 10

Bomb Plot. Sampling Experiment.

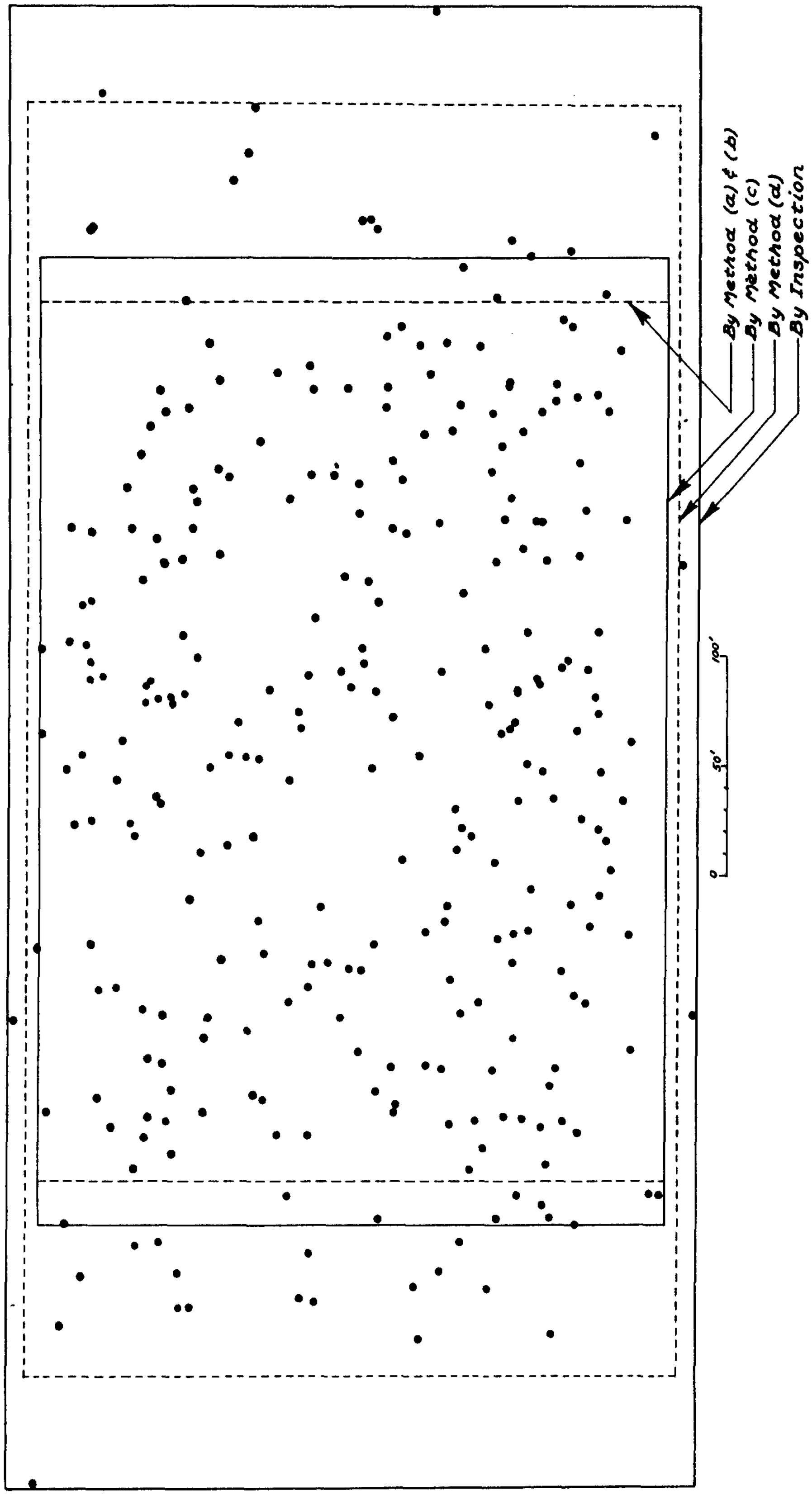

Figure 11

Bomb Plot. Sampling Experiment.

> P. 12. "Bombing with combat-box patterns means that individual bombs are not being aimed at individual structures at all. If a circle be drawn circumscribing the target complex, our efforts are really directed to placing a compact pattern of bombs in such a position that as many as possible come within this circle, depending upon chance and a high density to provide the hits on key structures."
>
> P. 21. "The relation of pattern size with the percent of bombs in a circle of 1000 foot radius about the aiming point is not immediately discernible. Small patterns well aimed will give large percentages, and poorly aimed may give zero percent. Large patterns cannot give high percentages even if well aimed but will tolerate greater aiming errors."

The above quotations combined with reports on recent operations in various theaters suggested three different problems of "optimum pattern".

(i) The problem of the bomb pattern yielding the greatest average percent of bombs falling within a square of given dimensions 2a x 2a about the aiming point*.

(ii) The problem of the pattern of bombs yielding the greatest probability of hitting a circular sub-target within the target area with reference to the particular location of this sub-target.

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*Presumably there will be no opposition to the change from a circle to a square, for which the algebra of the solution is easier.

Data on the optimum pattern of this kind may be useful in cases when a knockout attack on a target area is contemplated. In such cases the information desired is the number of planes and bombs that will insure a very high chance of hitting each and every sub-target. As this chance depends on the pattern of bombs released by each formation, it is natural to use the particular pattern which maximizes the chance of hitting the sub-target whose location makes it the most difficult to hit.

It may be useful to illustrate this kind of situation in two examples.

(1) When trying to clear a path through a mine field by bombing, it seems natural to make sure that all land mines, even those which are most difficult to destroy, have a high chance of being exploded by bombs.

(2) In preparing for recent landing operations on islands in the Pacific, obviously efforts were made to knock out entirely whatever aircraft there may have been on the islands and to destroy all major defense installations as well.

(iii) The problem of the pattern of bombs yielding the greatest probability of hitting a circular sub-target within the target area, the particular location of which is left unspecified.

The data on optimum patterns of this kind will be useful in what may be described as routine bombing. Suppose, for example, that several formations of bombers are dispatched each to attack

a different factory area or, perhaps, different air bases. The purpose of such attacks is to inflict as much damage as possible upon each target area, although no special effort is directed towards a complete knockout blow, since this may require the concentration of more than one formation on a single target area.

In circumstances of this kind there may be a disadvantage in making efforts to increase the probability of hitting the sub-target which is most difficult to hit, since this may decrease substantially the chances of hitting other sub-targets. What is wanted is to hit as many sub-targets within the given area as practicable, without insisting on any one in particular nor on a complete coverage, for which the force dispatched would be insufficient.

Missions of this kind are, probably, the most frequent of all and, at this moment, it does not seem necessary to give specific examples. Some illustrations will be found in subsection VII-D.

B. Pattern dimensions maximizing the average proportion of bombs within a square about the aiming point.

The formula

$$p_1 = \frac{\sigma_{a_d}}{A}\left[\Pi\left(\frac{a+A}{\sigma_{a_d}}\right) - \Pi\left(\frac{a-A}{\sigma_{a_d}}\right)\right] \cdot \frac{\sigma_{a_r}}{B}\left[\Pi\left(\frac{a+B}{\sigma_{a_r}}\right) - \Pi\left(\frac{a-B}{\sigma_{a_r}}\right)\right] \qquad (78)$$

which was deduced in section V-C gives the expected proportion of

bombs falling within a square 2a x 2a about the aiming point. Operationally the value of p_1 will be approximated by the <u>average proportion</u> of bombs within the square achieved in a number of similar missions. To obtain the pattern maximizing p_1 it is sufficient to find the value of A maximizing the factor

$$\phi(A) = \frac{\sigma_{a_d}}{A}\left[\Pi\left(\frac{a+A}{\sigma_{a_d}}\right) - \Pi\left(\frac{a-A}{\sigma_{a_d}}\right)\right] \tag{79}$$

and the value of B maximizing a similar expression representing the second factor in (78). We have

$$\frac{d\phi}{dA} = -\frac{1}{A^2}\Psi(A), \tag{80}$$

with

$$\Psi(A) = a\left[G\left(\frac{a+A}{\sigma_{a_d}}\right) - G\left(\frac{a-A}{\sigma_{a_d}}\right)\right] + \sigma_{a_d}\left[g\left(\frac{a+A}{\sigma_{a_d}}\right) - g\left(\frac{a-A}{\sigma_{a_d}}\right)\right]. \tag{81}$$

Further,

$$\frac{d\Psi}{dA} = \frac{A}{\sigma_{a_d}}\left[g\left(\frac{a-A}{\sigma_{a_d}}\right) - g\left(\frac{a+A}{\sigma_{a_d}}\right)\right] \geq 0. \tag{82}$$

Hence $\Psi(A)$ is a monotone increasing function of A. At $A = 0$ we have $\Psi(0) = 0$. Hence for $A > 0$ the function $\Psi(A)$ is positive. Therefore the derivative of $\phi(A)$ is zero at $A = 0$ and is negative for $A > 0$. It follows that the values of A and B maximizing p_1 are $A = B = 0$.

Therefore, in order to obtain the greatest <u>average</u> percent of bombs within the square 2a x 2a about the aiming point, it is necessary and sufficient to tend to make the dimensions of the bomb pattern as small as possible, whatever be the dimensions of the square target and whatever be the standard errors of aiming.

C. <u>Pattern dimensions maximizing the probability of hitting a circular sub-target of specified location</u>.

If it is accepted that formula (13), with (19),

$$P^{*}_{F,0} = \{1 - I(1-e^{-D})\}^{F} \tag{83}$$

represents the probability of missing a circle S(x,y) with sufficient accuracy, then the problem of the optimum pattern will be solved by selecting the values, say $\hat{A}$ and $\hat{B}$ of

$$A' = A/\sigma_{a_d} \text{ and } B' = B/\sigma_{a_r} \tag{84}$$

which will maximize the expression

$$\lambda(A', B') = I(1-e^{-D})$$
$$= \left[G(x'+A') - G(x'-A')\right]\left[G(y'+B') - G(y'-B')\right]\left[1-e^{-\frac{\pi\Delta}{A'B'}}\right] \tag{85}$$

with

$$x' = x/\sigma_{a_d},\ y' = y/\sigma_{a_r}$$
$$\Delta = \frac{mR^2}{4\,\sigma_{a_d}\sigma_{a_r}} \tag{86}$$

$\hat{A}$ and $\hat{B}$ will be called the standardized optimum half dimensions of the bomb pattern.

Obviously $\hat{A}$ and $\hat{B}$ are functions of three variables, x', y' and Δ. A sample of values of $\hat{A}$ and $\hat{B}$ corresponding to $\Delta = 1$ and to varying values of x' and y' is given in Table XIII. Section VII contains a nomogram and charts determining the values of $\hat{A}$ and $\hat{B}$.

Table XIII is divided into nine columns and nine double rows, each cell containing two numbers. The upper number is the value of $\hat{A}$ and the lower that of $\hat{B}$. It will be seen that the optimum values of A and B are not equal to zero and that they depend on m, R, σ_{a_d} and σ_{a_r}. At first sight this result may seem contradictory to the one obtained in subsection B, namely that whatever be m, R, σ_{a_d} and σ_{a_r}, to increase the average proportion of bombs falling within a square about the aiming point, one should diminish the pattern dimensions as far as practicable. However, the contradiction is only an apparent one. As the authors of the Report of the VIII B.C. rightly point out, if the dimensions of the bomb pattern are too small for the given values of the S.E.'s of aiming, then very frequently the whole pattern will be placed outside the target. Also, with very small patterns there will be many overlaps of craters which, from the point of view of at least one hit, constitutes a waste of bombs. On the other hand it happens that the <u>average</u> per cent of hits is greatest

TABLE XIII

Standardized Half Dimensions $\hat{A}$ and $\hat{B}$ of Bomb Pattern, Optimum for Hitting a Circular Sub-Target S(x,y) of Known Location

$\Delta = 1.0$

y \ x		0	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
0	$\hat{A}$	1.5	1.7	2.1	2.7	3.3	3.8	4.4	5.0	5.5
	$\hat{B}$	1.5	1.4	1.3	1.2	1.2	1.1	1.0	1.0	1.0
.5	$\hat{A}$	1.4	1.6	2.0	2.7	3.3	3.8	4.4	4.9	5.5
	$\hat{B}$	1.7	1.6	1.4	1.3	1.2	1.2	1.1	1.1	1.0
1.0	$\hat{A}$	1.3	1.4	1.9	2.6	3.2	3.8	4.3	4.9	5.4
	$\hat{B}$	2.1	2.0	1.9	1.8	1.7	1.6	1.6	1.5	1.5
1.5	$\hat{A}$	1.2	1.3	1.8	2.5	3.1	3.7	4.3	4.8	5.4
	$\hat{B}$	2.7	2.7	2.6	2.5	2.4	2.3	2.3	2.3	2.2
2.0	$\hat{A}$	1.2	1.2	1.7	2.4	3.0	3.6	4.2	4.8	5.4
	$\hat{B}$	3.3	3.3	3.2	3.1	3.0	3.0	3.0	3.0	3.0
2.5	$\hat{A}$	1.1	1.2	1.6	2.3	3.0	3.6	4.2	4.8	5.4
	$\hat{B}$	3.8	3.8	3.8	3.7	3.6	3.6	3.6	3.6	3.6
3.0	$\hat{A}$	1.0	1.1	1.6	2.3	3.0	3.6	4.2	4.8	5.4
	$\hat{B}$	4.4	4.4	4.3	4.3	4.2	4.2	4.2	4.2	4.2
3.5	$\hat{A}$	1.0	1.1	1.5	2.3	3.0	3.6	4.2	4.7	5.3
	$\hat{B}$	5.0	4.9	4.9	4.8	4.8	4.8	4.8	4.7	4.7
4.0	$\hat{A}$	1.0	1.0	1.5	2.2	3.0	3.6	4.2	4.7	5.3
	$\hat{B}$	5.5	5.5	5.4	5.4	5.4	5.4	5.4	5.3	5.3

TABLE XIV

Standardized Half Dimensions A* and B* of Bomb Pattern Maximizing the Proportion of Circular Sub-Targets Within the Target Area, Which Will be Hit in One Formation Attack

$\Delta = 1.0$

y \ x		0	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
0	A*	1.5	1.5	1.7	1.9	2.2	2.5	2.9	3.3	3.7
	B*	1.5	1.5	1.4	1.4	1.3	1.3	1.2	1.1	1.1
.5	A*	1.5	1.5	1.7	1.9	2.2	2.5	2.9	3.3	3.7
	B*	1.5	1.5	1.5	1.4	1.3	1.3	1.2	1.2	1.1
1.0	A*	1.4	1.5	1.6	1.8	2.1	2.4	2.8	3.2	3.6
	B*	1.7	1.7	1.6	1.6	1.5	1.4	1.4	1.3	1.2
1.5	A*	1.4	1.4	1.6	1.8	2.1	2.4	2.7	3.1	3.5
	B*	1.9	1.9	1.8	1.8	1.7	1.6	1.5	1.5	1.4
2.0	A*	1.3	1.3	1.5	1.7	2.0	2.3	2.6	3.0	3.4
	B*	2.2	2.2	2.1	2.1	2.0	1.9	1.8	1.7	1.7
2.5	A*	1.3	1.3	1.4	1.6	1.9	2.2	2.6	2.9	3.3
	B*	2.5	2.5	2.4	2.4	2.3	2.2	2.1	2.1	2.0
3.0	A*	1.2	1.2	1.4	1.5	1.8	2.1	2.5	2.8	3.2
	B*	2.9	2.9	2.8	2.7	2.6	2.6	2.5	2.4	2.4
3.5	A*	1.1	1.2	1.3	1.5	1.7	2.1	2.4	2.8	3.2
	B*	3.3	3.3	3.2	3.1	3.0	2.9	2.8	2.8	2.7
4.0	A*	1.1	1.1	1.2	1.4	1.7	2.0	2.4	2.7	3.1
	B*	3.7	3.7	3.6	3.5	3.4	3.3	3.2	3.2	3.1

when the dimensions of the pattern are equal to zero. This greatest value would be attained by averaging a considerable number of zeros with a few cases of 100 percent hits.

Once the optimum values $\hat{A}$ and $\hat{B}$ are obtained, the solution of equations (76) gives the optimum values of d and u, namely

$$d_{opt.} = \sqrt{\frac{\hat{A}^2 \sigma_{a_d}^2 - 3(\sigma_f^2 + \sigma_{d_d}^2)}{s^2 - 1}} \qquad (87)$$

$$u_{opt.} = \sqrt{\frac{\hat{B}^2 \sigma_{a_r}^2 - 3(\sigma_F^2 + \sigma_{d_r}^2)}{n^2 - 1}} \qquad (88)$$

While no inconsistencies were found with respect to the formula giving the optimum value of u, the authors feel it necessary to caution one concerning the use of formula (87). It must be remembered that the basic assumption of Robertson's theory is that all the C-bombs are uniformly distributed within the bomb pattern. The formulae, as it were, take this circumstance for granted and, if the assumption of uniformity is strongly violated, cannot help leading to incorrect results.

Suppose for example (this is exactly what happened in a particular problem) that for a three string formation of 18 planes it is found that $d_{opt.} = 350'$. This suggests that the three strings of planes should be spaced laterally 700' from one string to another. The more detailed theory applied to the same

problem indicated that such a wide spacing is not advantageous. The authors presume this to be the fact and the reason for the discrepancy is that, if the three strings of bombers are as widely spaced as indicated by $d_{opt.}$, then the distribution of bombs within the pattern will be far from uniform. In fact, rather than uniform it will be composed of three parallel ridges with furrows in between.

An interpretation of the result which seems logical to the authors is that with a large value of A, instead of flying a three string formation, a pattern of some six or even nine planes abreast (or in "Vee's) would be indicated so as to insure a greater homogeneity in the density of bombs all over the intended pattern. This conclusion is confirmed by the values of both $\mathcal{P}_{F,0}$ and $P^*_{F,0}$.

Inspection of formulae (83) and (85) shows that for fixed values of Δ, A', and B', the probability $P^*_{F,0}$ is an increasing function of both $|x'|$ and $|y'|$. Therefore, (within a rectangular target area) the subtarget S(x,y) which is most difficult to hit is the one that is placed in the corner of the target area. It follows that when planning a knockout attack on a rectangular target area with the aiming point at its center, one should use the optimum pattern dimensions $2\hat{A}$ and $2\hat{B}$ corresponding to the coordinates (x,y) of the corner of the target and should send a

force sufficient to insure an adequately large probability of hitting a circular sub-target in the corner. Then the probability of destroying other sub-targets will be even more satisfactory.

D. <u>Pattern dimensions maximizing the expected number of circular sub-targets hit in a single formation attack.</u>

Suppose that a rectangular target area $2\xi \times 2\eta$ about the aiming point contains a certain number of circular sub-targets S of radius R, distributed over the area at random. For example, the target may be a runway together with the dispersal area surrounding the runway. In this case, the sub-targets would be the parked aircraft. Since the parking places may be expected to vary from day to day, the correspondence with the above assumption is complete.

The problem, considered in this sub-section, is to determine the dimensions of the bomb pattern which maximize the probability that a single formation attacking the area will hit the target S. If the dimensions of the bomb pattern are 2A x 2B then this probability will be denoted by $Q(\xi,\eta,A,B)$. It is obvious that the value of $Q(\xi,\eta,A,B)$ also represents the expected proportion of targets S distributed at random over the target area, which will be hit at least once. Also it is approximately equal to the expected proportion of sub-targets hit if the

sub-targets have a fixed location in the target area but are more or less uniformly distributed over it.

Writing $\xi = \xi' \sigma_{a_d}$, $\eta = \eta' \sigma_{a_r}$ it is easily found that

$$Q(\xi, \eta, A, B)$$
$$= \left(1 - e^{-\frac{\pi A}{A'B'}}\right) \cdot \frac{1}{\xi'} \int_0^{\xi'} \{G(x'+A') - G(x'-A')\} dx \cdot \frac{1}{\eta'} \int_0^{\eta'} \{G(y'+B') - G(y'-B')\} dy$$
$$= \left(1 - e^{-\frac{\pi A}{A'B'}}\right) \cdot \{\Pi(\xi'+A') - \Pi(\xi'-A')\} \cdot \{\Pi(\eta'+B') - \Pi(\eta'-B')\} / \xi'\eta' \qquad (89)$$

The values of A' and B' maximizing this expression will be denoted by A^* and B^*. Table XIV, perfectly analogous to Table XIII, gives a sample of values of A^* and B^*. Comparison of Tables XIII and XIV shows that generally the pattern dimensions maximizing the probability of hitting a circular sub-target of known location differ from those most advantageous for hitting a target placed in the target area at random. Illustrations given in section VII indicate that at least in certain cases these differences are reflected in substantial differences in the probabilities. Section VII also contains a nomogram for computing the values of A^* and B^*.

VII. USE OF GRAPHS AND SUMMARY OF ARITHMETICAL PROCEDURES INVOLVED IN PRACTICAL APPLICATIONS OF THE THEORY OF ROBERTSON'S HYPOTHESIS. ILLUSTRATIONS.

A. General. The results of Robertson's theory of

formation bombing which, at the present time, seem the most useful are

(i) The formula giving the expected proportion of bombs released by a formation, falling within a rectangle;

(ii) The formula for the standard error of the proportion of bombs falling within a given rectangle;

(iii) Means of computing the dimensions of the rectangular bomb pattern (and thereby the lateral spacing of planes and the spacing of bombs in train) which maximize the probability of hitting a circular sub-target $S(x,y)$ placed at a point (x,y);

(iv) Means of computing the dimensions of the rectangular bomb pattern (and thereby the lateral spacing of planes and the spacing of bombs in train) which maximize the expected proportion of circular sub-targets distributed within a rectangle $2\xi \times 2\eta$ about the aiming point which will be hit in one formation attack;

(v) The formula giving the probability of hitting (or missing) a circular sub-target $S(x,y)$ with its center at a specified point (x,y);

(vi) The formula giving the expected number of circular targets distributed within a rectangle $2\xi \times 2\eta$ about the aiming point, which will be hit in one formation attack.

The purpose of this section is to summarize the arithmetical procedures involved, to explain the details of the graphical

methods by which these procedures can be simplified and to illustrate the results on a few examples.

The solution of all the problems discussed depends on the standard errors of aiming $\sigma_{a_r}, \sigma_{a_d}$, on the S.E.'s of formation pattern σ_F, σ_f and on the S.E.'s of dispersion of bombs σ_{d_r}, σ_{d_d} in range and deflection respectively. Therefore it seems appropriate to summarize the information available to the authors on the values of these parameters.

The values of σ_{a_r} and σ_{a_d} reported from various theaters are extremely variable and range from several hundred feet to almost 2000'feet. Also, as the Operations Analysis Sections appear to be estimating the precision of aiming, no particular assumption as to the values of σ_{a_r} and σ_{a_d} is suggested.

The information available on the standard errors of formation pattern is limited to three bomb plots obtained in experimental bombing and to the plots obtained in combat, published in the Report dated March 31, 1944 of the Operations Analysis Section of the XV Air Force. Both sources give surprisingly consistent estimates of about

$$\sigma_f = 100' \quad \text{and} \quad \sigma_F = 500'.$$

The standard errors of bomb dispersion are small in comparison with the other S.E.'s and, since they always appear in sums of the type $\sigma_f^2 + \sigma_{d_d}^2$ or $\sigma_{a_d}^2 + \sigma_f^2 + \sigma_{d_d}^2$ in the computations

of this report, their importance is only moderate. With no fear of serious errors the following rough estimates may be used:

G.P. bombs up to and including 500 lb. .... $\sigma_{d_d} = \sigma_{d_r} = 40'$

1000 lb. bombs released from over 12,000' altitude .... $\sigma_{d_d} = \sigma_{d_r} = 150'$

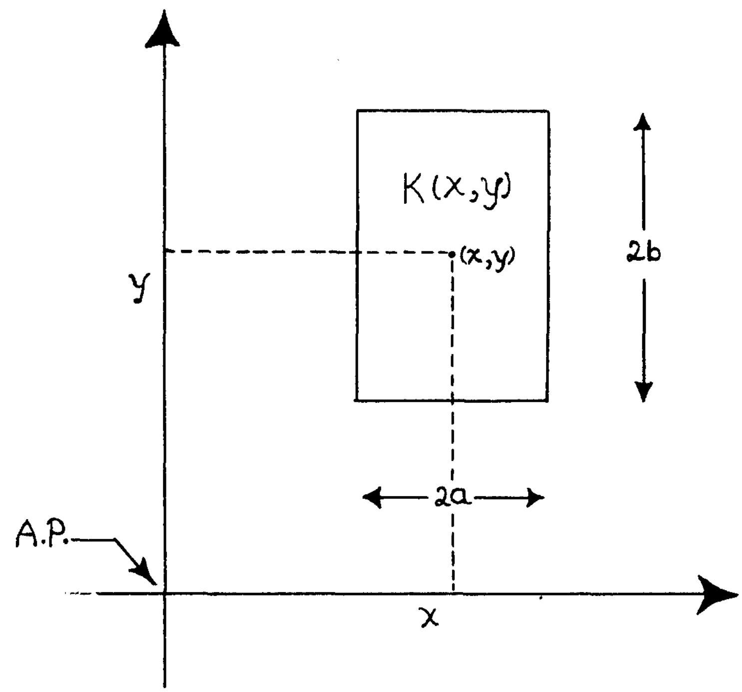
20 lb. fragmentation bombs .... $\sigma_{d_d} = \sigma_{d_r} = 100'$

The empirical data available concerning dispersion were discussed more fully in the First Cooperative Study. Here it will suffice to state that most of the data refer to practice bombs, that considerably less is known on the lighter G.P. bombs and that the information available to the authors on the 1000 lb. bombs as well as on the 20 lb. fragmentation bombs is very meager. The figures given above must be interpreted having in view the scarcity of data on which they are based.

The above values of the standard errors of formation pattern and of bomb dispersion will be used in all the examples given below.

B. Proportion of bombs hitting a rectangle; its expectation and its standard error.

Let $K(x,y)$ denote a rectangle of dimensions $2a \times 2b$, the side $2b$ being parallel to the line of flight. The rectangle $K(x,y)$ is centered at the point (x,y) with respect to axes passing through the aiming point.



If F formations aim at the origin of coordinates, each releasing a rectangular pattern of bombs of the same dimensions 2A x 2B, then the expected proportion p_1 of bombs hitting the rectangle K(x,y) is given by the formula

$$p_1 = \frac{\sigma_x}{2A}\{\Pi(\frac{x+a+A}{\sigma_x}) - \Pi(\frac{x-a+A}{\sigma_x}) - \Pi(\frac{x+a-A}{\sigma_x}) + \Pi(\frac{x-a-A}{\sigma_x})\} \cdot$$

$$\cdot \frac{\sigma_y}{2B}\{\Pi(\frac{y+b+B}{\sigma_y}) - \Pi(\frac{y-b+B}{\sigma_y}) - \Pi(\frac{y+b-B}{\sigma_y}) + \Pi(\frac{y-b-B}{\sigma_y})\} \qquad (90)$$

where

$$\sigma_x = \sqrt{\sigma^2_{a_d} + \sigma^2_f + \sigma^2_{d_d}}$$

$$\sigma_y = \sqrt{\sigma^2_{a_r} + \sigma^2_F + \sigma^2_{d_r}} \qquad (91)$$

When the dimensions of the bomb pattern are unknown and it is desired to estimate them from the structure of the formation, the following procedure is recommended.

If the formation is composed of s strings of planes, and if the spacing between the strings is 2d feet, then

$$A = sd. \tag{92}$$

Similarly, if n denotes the number of bombs in each train which are released singly (not in clusters) by the intervalometer and if the spacing between the bombs is 2u, then

$$B = nu. \tag{93}$$

In the case of clusters of fragmentation bombs, each cluster plays the role of a single bomb. Thus if the load contains 144 fragmentation bombs forming 24 clusters of 6 bombs each, then $n = 24$.

Remark: <u>The formulae $A = sd$ and $B = nu$ are applicable in the present problem</u> and in the following one concerned with the value of σ_H <u>but not in the problem of computing $\overset{*}{P}_{F,0}$</u>.

If in a particular case there is only one string of planes then $A = 0$. In this case the first factor in the formula for p_1 reduces to

$$G\left(\frac{x+a}{\sigma_x}\right) - G\left(\frac{x-a}{\sigma_x}\right) \tag{94}$$

Similarly, if $u = 0$ and, therefore, $B = 0$, the second factor of the formula for p_1 reduces to

$$G(\frac{y+b}{\sigma_y}) - G(\frac{y-b}{\sigma_y}) \tag{95}$$

It will be seen that the computation of p_1 requires (i) some source of values of the functions $G(x)$ and $\Pi(x)$ and (ii) elementary arithmetical operations which can easily be performed with a slide rule.

Charts I and II are constructed to provide easy means of obtaining the values of $G(x)$ and $\Pi(x)$.

Example 1. Suppose that a formation composed of two squadrons of $N = 9$ planes each attacks a runway 400' x 4000' flying along its axis. Each of the squadrons flies a three string formation with lateral spacing $2d = 110'$. The leading plane of the first squadron aims the center of its train at the center of the runway. The second squadron is staggered to the right, the lateral distance between the central planes of the two squadrons being 820'. All the planes of both squadrons release their bombs on the leader of the first squadron. It is assumed that the unavoidable lag in time of bomb release tends to compensate the stagger and that, on the average, the range of all trains of bombs is the same. The number of bombs released by each plane is $n = 12$ and they are spaced $2u = 360'$ apart.

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt$$

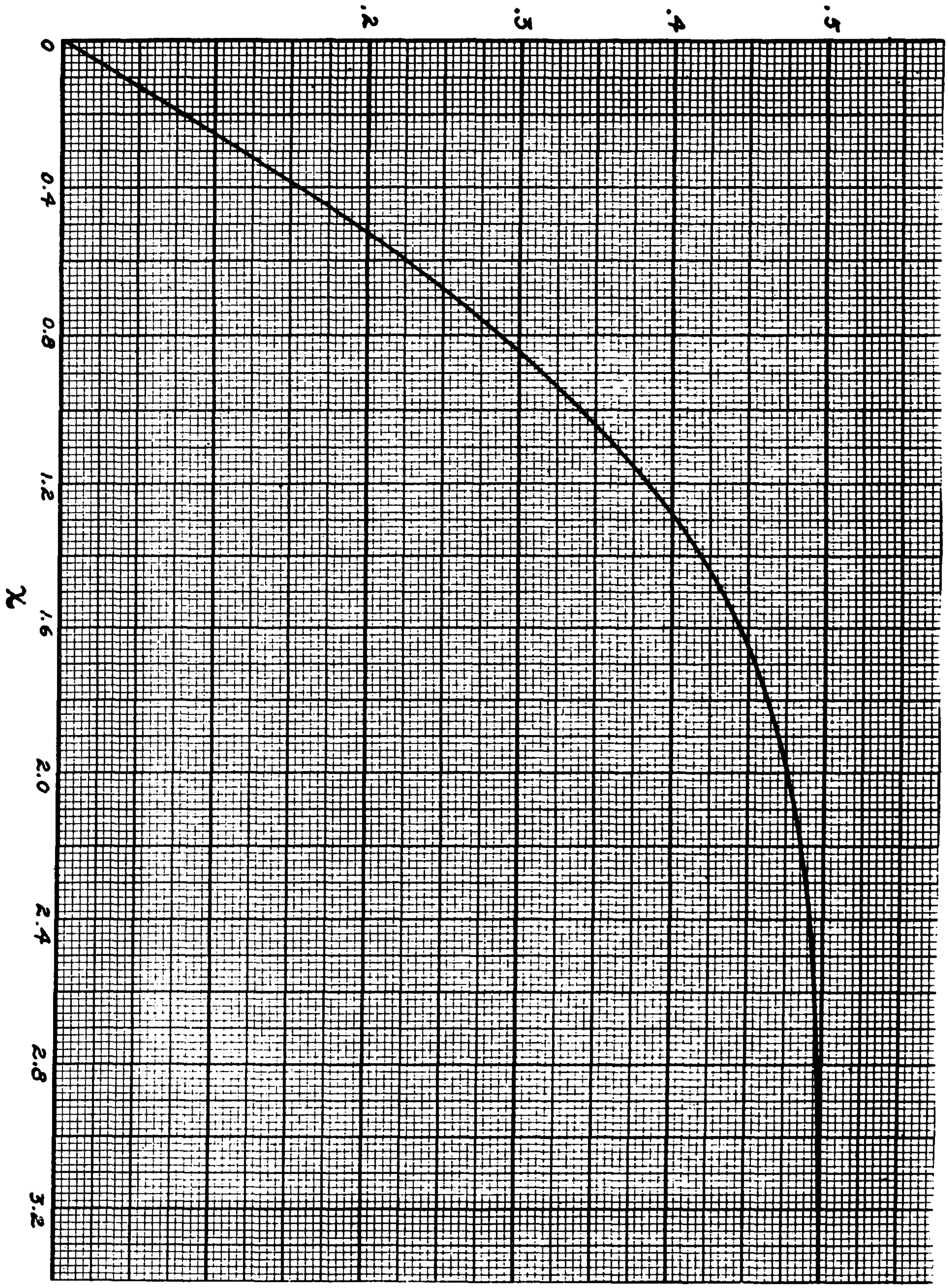


Chart I

$$\Pi(x) = x\,G(x) + g(x) = \frac{x}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}\,dt + \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}$$

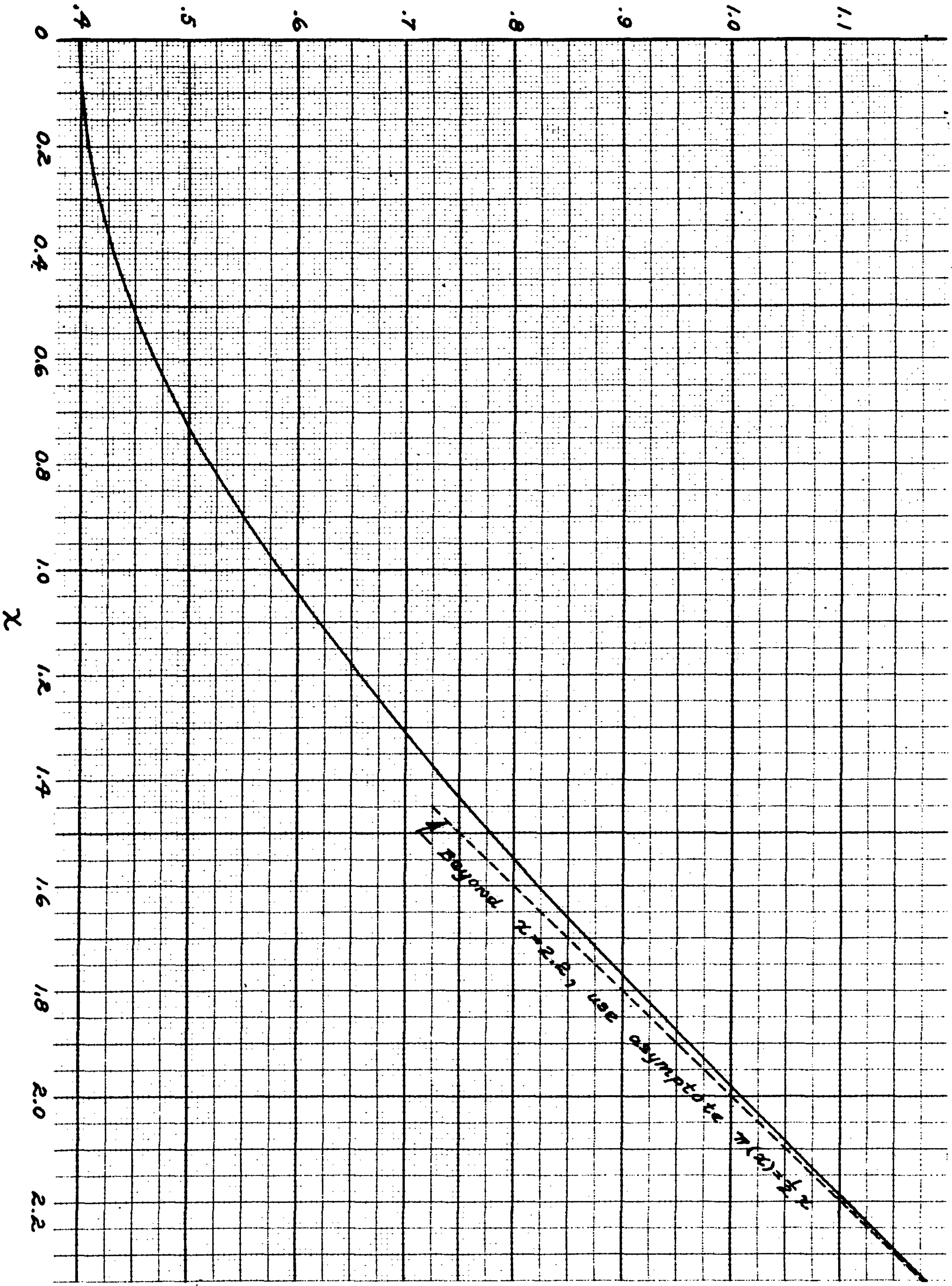


Chart II

The problem which will be used to illustrate the method of computation is that of determining what proportions, p_1' and p_1'', of bombs released by two squadrons may be expected to fall on the runway. It will be assumed* $\sigma_x = 400'$, $\sigma_y = 1600'$. We begin by

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*The value 820' of the lateral spacing between the two squadrons and the values $\sigma_x = 400'$ and $\sigma_y = 1600'$ were found to be one of several combinations which fitted the distribution of the average percent of bomb fall published by the Operations Analysis Section of the Thirteenth Air Force in their Preliminary Reports No. 9 and No. 11. A sample of the computations made to obtain rough estimates of these three parameters is as follows.

Region:			1	2	3	4	5
Observed Percent of Bombs:			16	32	24	13	15
Lateral Distance between Leaders	σ_x	σ_y	Expected Percent of Bombs				
270	1000	1000	11.8	32.3	27.2	16.8	11.8
	800	1600	12.4	33.5	26.5	15.1	12.5
520	900	900	12.9	34.2	27.8	15.7	9.5
	700	1400	14.1	36.3	26.8	13.7	9.1
670	800	800	13.8	36.4	27.5	14.5	7.8
	600	1600	13.9	35.8	25.6	13.4	11.3
	400	2000	16.1	36.2	21.8	11.1	14.9
820	600	600	16.2	41.4	25.8	11.3	5.3
	600	1000	14.8	38.7	26.4	13.1	7.0
	600	1400	13.6	35.6	25.9	14.2	10.7
	700	1200	13.1	34.6	26.8	15.2	10.4
	400	1600	16.8	36.8	23.3	12.2	11.0

The combinations of values of the parameters given in the last three lines fit the observational data about as well as any of the others. Owing to the particular form of data available and to the rough method of estimation, these estimates are not reliable but may be used for purposes of illustration.

computing the arguments of the formula for p_1. For each squadron we have

$$s = 3, \quad d = 55, \quad n = 12, \quad u = 180.$$

Hence

$$A = sd = 165, \quad B = nu = 2160,$$

Also, for both squadrons,

$$a = 200, \quad b = 2000.$$

It is convenient to express these values in terms of the σ's. This also applies to the value of $x = -820$ which is needed for the computations concerning the second squadron. We have

$$A/\sigma_x = .4125, \qquad B/\sigma_y = 1.35,$$

$$a/\sigma_x = .5, \quad b/\sigma_y = 1.25, \quad x/\sigma_x = -2.05.$$

The second factor in the formula for p_1 has the same value for both squadrons. Since $y = 0$, the two positive terms as well as the two negative terms within the bracket become equal. Thus the computations reduce to evaluating one positive term and one negative term and to dividing the difference by B/σ_y instead of by $2B/\sigma_y$.

Arguments of Π	Corresponding Π from Chart II
$(b+B)/\sigma_y = 2.60$	1.301
$(b-B)/\sigma_y = -.10$	.401
Difference	.900
Ratio of the last difference to B/σ_y	.667

The computation of the first factor in the formula for p_1 must be done separately for each of the two squadrons. For the first squadron, since x = 0, the computations follow the pattern described above for y = 0. However, for the second squadron, $x/\sigma_x = -2.05$ and therefore the first bracket of the formula for p_1 contains all four terms. The divisor in this case is $2A/\sigma_x$.

First Squadron

Arguments of Π	Corresponding Values of Π from Chart II
$(a+A)/\sigma_x = .9125$	.554
$(a-A)/\sigma_x = .0875$	.400
Difference	.154
Ratio of the difference to A/σ_x	.373

Second Squadron

Arguments of Π	Corresponding Values of Π from Chart II	
$(x+a+A)/\sigma_x = -1.1375$	.632	
$(x-a-A)/\sigma_x = -2.9625$	1.482	
Sum		2.114
$(x-a+A)/\sigma_x = -2.1375$	1.075	
$(x+a-A)/\sigma_x = -1.9625$	.991	
Sum		2.066
Difference		0.048
Ratio of the difference to $2A/\sigma_x$		.058

Thus the values of p_1 for both squadrons are

First Squadron

$$p_1' = .373 \times .667 = .249$$

Second Squadron

$$p_1'' = .058 \times .667 = .039$$

The formula for the standard error σ_H of the proportion of bombs hitting the rectangle K(x,y) depends on both the number m of bombs released by each of the attacking formations and on

the number F of formations.

$$\sigma_H = \left\{\frac{p_1(1-p_1)}{Fm} + \frac{m-1}{Fm}(p_2-p_1^2)\right\}^{\frac{1}{2}} \tag{96}$$

Here p_1 denotes the expected proportion of bombs hitting the rectangle $K(x,y)$ (computed above) and p_2 the probability of two specified bombs released by the same formation hitting $K(x,y)$ and $m = Nn$ the total number of all bombs dropped by a given squadron. The formula for computing p_2 is a product of the two values of the same function

$$p_2 = f(x,a,A,\sigma_x)f(y,b,B,\sigma_y) \tag{97}$$

The function f has two forms depending on whether the second of its arguments is greater than the third or vice versa. For example, if $a \leqq A$ then

$$\begin{aligned} f(x,a,A,\sigma_x) = \{ & \frac{x+a+A}{\sigma_x}\Pi(\frac{x+a+A}{\sigma_x}) + G(\frac{x+a+A}{\sigma_x}) \\ & + \frac{x-3a-A}{\sigma_x}\Pi(\frac{x+a-A}{\sigma_x}) + G(\frac{x+a-A}{\sigma_x}) \\ & - \frac{x+3a+A}{\sigma_x}\Pi(\frac{x-a+A}{\sigma_x}) - G(\frac{x-a+A}{\sigma_x}) \\ & - \frac{x-a-A}{\sigma_x}\Pi(\frac{x-a-A}{\sigma_x}) - G(\frac{x-a-A}{\sigma_x})\}(\frac{\sigma_x}{2A})^2 \\ = {} & f_1 \text{ (say)} \end{aligned} \tag{98}$$

On the other hand, if $A \leqq a$ then

$$f(x,a,A,\sigma_x) = \{\frac{x+a+A}{\sigma_x}\Pi(\frac{x+a+A}{\sigma_x}) + G(\frac{x+a+A}{\sigma_x})$$
$$+ \frac{x-a-3A}{\sigma_x}\Pi(\frac{x-a+A}{\sigma_x}) + G(\frac{x-a+A}{\sigma_x})$$
$$- \frac{x+a+3A}{\sigma_x}\Pi(\frac{x+a-A}{\sigma_x}) - G(\frac{x+a-A}{\sigma_x})$$
$$- \frac{x-a-A}{\sigma_x}\Pi(\frac{x-a-A}{\sigma_x}) - G(\frac{x-a-A}{\sigma_x})\}(\frac{\sigma_x}{2A})^2$$
$$= f_2 \quad \text{(say)}. \tag{99}$$

In computing p_2 it is essential to keep in mind the distinction between these two cases. It may well happen that the first factor of the product giving p_2 has to be computed from formula f_1 and the second from formula f_2 (or vice versa). This would occur if $a < A$ but $B < b$. Otherwise the computations are simple and follow the same general pattern as those leading to the value of p_1.

<u>Example 2</u>. In the conditions of Example 1 the following values of p_2 are found:

First Squadron	Second Squadron
$p_2 = .137$	$p_2 = .039$

Assuming that each plane in the formation releases $n = 12$ bombs and that there are $N = 9$ planes per formation, the total number of bombs released by each squadron is $m = nN = 108$. Substituting these values in the formula for σ_H, with $F = 1, 2, 3$, the following

values are found:

First Squadron

p_1	F	σ_H
.249	1	.275
	2	.195
	3	.159

Second Squadron

ρ_1	F	σ_H
.039	1	.195
	2	.138
	3	.112

It is seen that the values of σ_H are rather large compared with those of the corresponding p_1. This indicates that, even though for the first squadron the expected proportion of bombs hitting the runway is fairly large, the variability in the actual number of hits scored in successive missions is very large. It follows that the average proportion of 24.9 percent must be the outcome of a number of totally unsuccessful missions, in which the number of hits on the runway is very few or zero, and of a few missions in which practically the whole pattern hits the runway. If this variability in the outcome of particular missions is found to be unsatisfactory, then the situation may be improved by flying wider patterns. By doing so the expected proportion of bombs falling on the runway will diminish somewhat, but the variability in this proportion will diminish very markedly, indicating that most of the missions will result in some damage to the runway.

For example, it is found that the results of attacks

made by single squadrons of N = 9 planes would be much more stable if, instead of flying a three string formation of three consecutive "Vee's", all the nine planes flew a single "Vee" with the lateral spacing between planes of 2d = 250', the spacing of bombs (n = 12 bombs in each train) being 2u = 360' as originally.

This particular pattern seems especially convenient, because its advantages persist for two alternative hypotheses concerning the standard errors of aiming which are likely to bracket the true values. The results of computations are given in Table XV.

TABLE XV

Frequency Constants Referring to a Nine String Formation Attacking a Runway 400' x 4000'.

2d = 250'	2u = 360'	
Hypothesis	σ_x = 400', σ_y = 1600'	σ_x = 700', σ_y = 1200'
p_1	.118	.117
σ_H	.050	.055
Probability of Missing Central 1000' of Runway	.10	.11

It will be seen that with this particular pattern σ_H is less than one half of the value of p_1 which indicates that

the actual proportion of bombs hitting the runway will be equal to zero only in relatively rare cases. This conclusion is reinforced by the third line of figures in Table XV representing the probability* that, with the particular pattern just described, the central area of the runway, 1000' feet long about the center, will be missed by all bombs of the nine plane formation. With both hypotheses concerning the standard errors of aiming it will be seen that in about nine missions out of ten there will be some craters within this central area. These craters are likely to bring about considerable difficulties in using the runway. On the other hand, the original pattern of the three string formation with 2d = 110' gives the value .31 to this probability*. It seems, then, that the increased chances of hitting the central section of the runway represent a considerable advantage for the wider formation pattern, outweighing the loss in the average proportion of bombs hitting the runway.

This example suggests the general idea that it is difficult to assess a given method of bombing by considering only one criterion such as, for example, the expected average number of bombs hitting a given area. The relative advantages or disadvantages become clearer if this expected average is supplemented by other data such as σ_H and/or the probability of missing.

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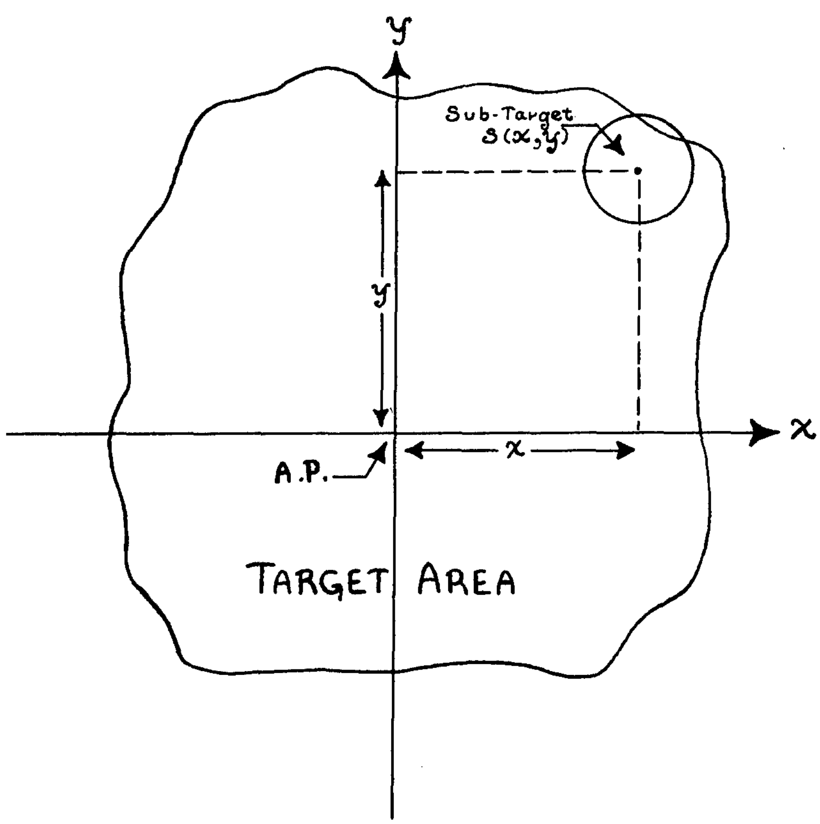
*These probabilities are computed from a formula not included in this report.

C. <u>The probability of hitting a circular sub-target. Optimum patterns</u>.

This subsection summarizes the results obtained relating to chances of hitting a circular sub-target of radius R, either placed at a specified point (x,y), probably in the corner of the target area where it is most difficult to hit, or placed at random within the target area.

Figure 12



The first of these problems will arise when a knockout attack is being planned and when the force to be sent is sufficient to insure a high chance of hitting all sub-targets, even those farthest from the aiming point. The situation is illustrated in Figure 12. The standardized (measured in terms of σ_{a_d} and σ_{a_r}) half dimensions of the bomb pattern which are optimum in this problem are denoted by $\hat{A}$ and $\hat{B}$.

The second problem arises when a formation of fixed size is sent to bomb a rectangular target area which is filled, more or less uniformly, with sub-targets or an area on which the sub-targets are distributed at random (parked planes on a dispersal area). It is desired to insure that the expected number of sub-targets hit is greatest.

The standardized half dimensions of the bomb pattern which are optimum in this second problem are denoted by A^* and B^* respectively.

In both cases the optimum half dimensions $\hat{A}$, $\hat{B}$ and A^*, B^* are obtained by the same method from similar nomograms given at the end of this report in Charts III and IV respectively. Therefore it is convenient to describe the process only once. Let x and y denote the coordinates of either (i) a specified sub-target for which it is desired to find $\hat{A}$ or $\hat{B}$; or (ii) the corner of a rectangular target with aiming point at its center for which it is desired to find A^* or B^*.

Let

$$x' = x/\sigma_{a_d}, \quad y' = y/\sigma_{a_r} \tag{100}$$

be the "standardized" values of these coordinates.

All four values $\hat{A}$, $\hat{B}$ and A^*, B^* are functions of x', y' and of the third variable

$$\Delta = \frac{mR^2}{4\sigma_{a_d}\sigma_{a_r}} \tag{101}$$

where m is the total number of bombs dropped by each formation.

In Charts III and IV the scale of Δ extends upward on either side of the nomogram. The two horizontal scales at the bottom are those of $\hat{A}$ and $\hat{B}$ in Chart III and of A^* and B^* in Chart IV. In each case the positive direction of the A scale is to the left and that of the B scâle to the right. Each of the curves fanning out to the left corresponds to a fixed value of x'. Each of those fanning out to the right corresponds to a fixed value of y'.

To read the optimum half dimensions of the bomb pattern corresponding to a given combination of values of x', y' and Δ, the method suggested is the use of a 45° right triangle placed on the appropriate nomogram with one side of the right angle vertical and the other horizontal. The vertex of the triangle must be kept on the horizontal line corresponding to the given value of Δ. Denote by u the intersection of the hypotenuse with the selected y' (or x') curve. Denote by v the intersection of the

vertical side of the triangle with the selected x' (or y') curve. Now by shifting the triangle right and left (always keeping the vertex on the horizontal Δ line), the position is found where the line connecting u and v is horizontal. The vertical lines through u and v when the line uv is horizontal determine the optimum standardized half dimensions of the bomb pattern. The key drawn on the nomogram illustrates the procedure.

Although the use of both Charts III and IV is straightforward the authors feel that it may be simplified. Figures 13 through 19 represent an attempt in this direction made in respect to $\hat{A}$ and $\hat{B}$. Each diagram corresponds to a fixed value of Δ. The axes of coordinates are those of x' and y'. The curves divide the quadrant of positive x' and y' into bands of approximately constant <u>minimum</u> probability of missing a sub-target S(x,y) by all the bombs released by a single formation. The figures within the quadrant give the values of $\hat{A}$ and $\hat{B}$ corresponding to points on which they are written. Of the two figures, the upper is the value of $\hat{A}$ and the lower that of $\hat{B}$. The purpose of diagrams 13 to 19 is not only to give the approximate values of $\hat{A}$ and $\hat{B}$ for a given system of x', y' and Δ but also to give a quick answer to the question of the number of formations necessary to reduce the optimum probability of missing a sub-target S(x,y) to the arbitrarily selected level of .15. The first of the figures near the end of the curves is the value of the <u>minimum</u> $P^{*}_{1,0}$ corresponding

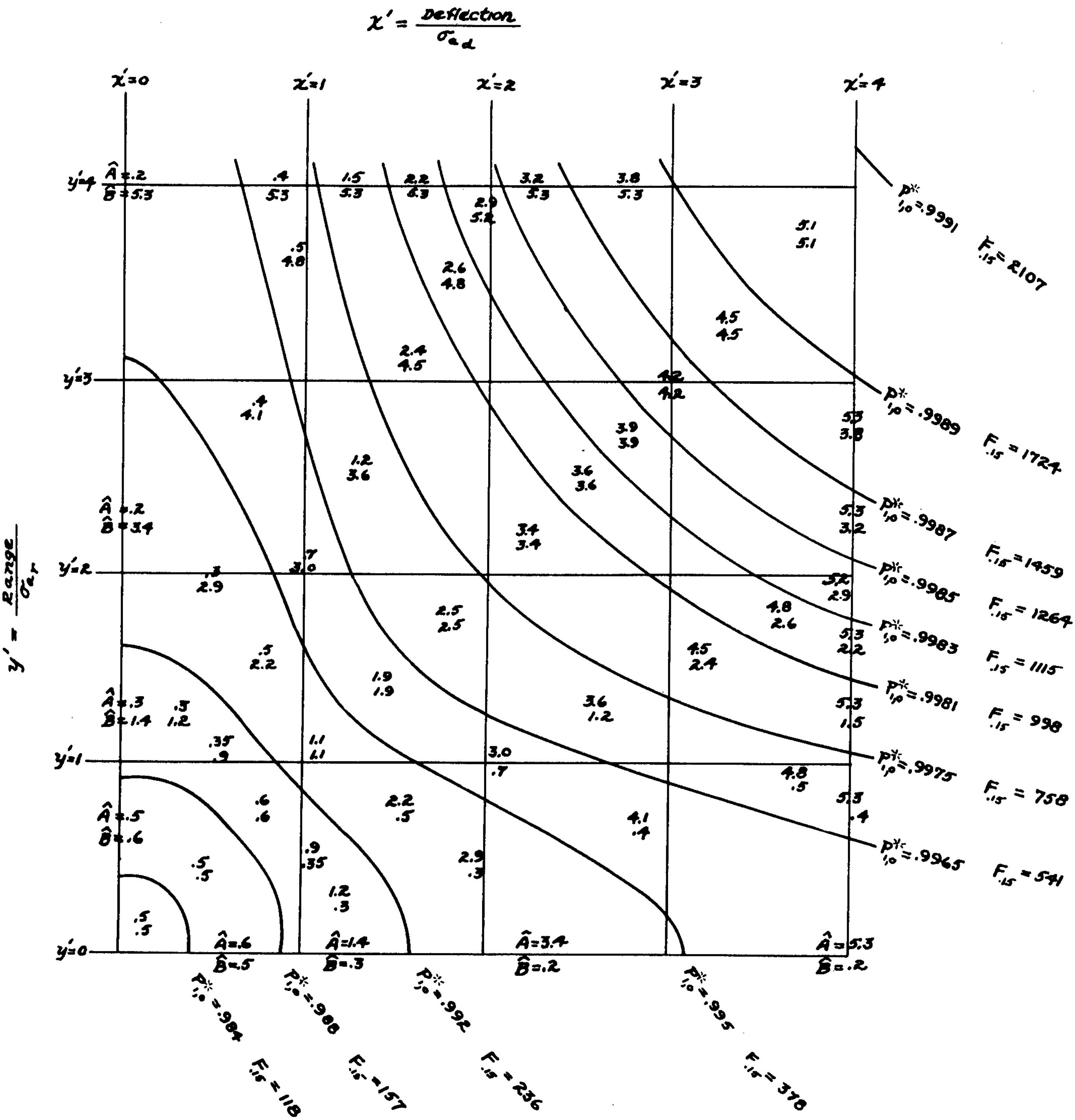


Figure 13

Curves of Optimum $P^*_{F,0}$. Values of $\hat{A}$ and $\hat{B}$.

$\Delta = .01$

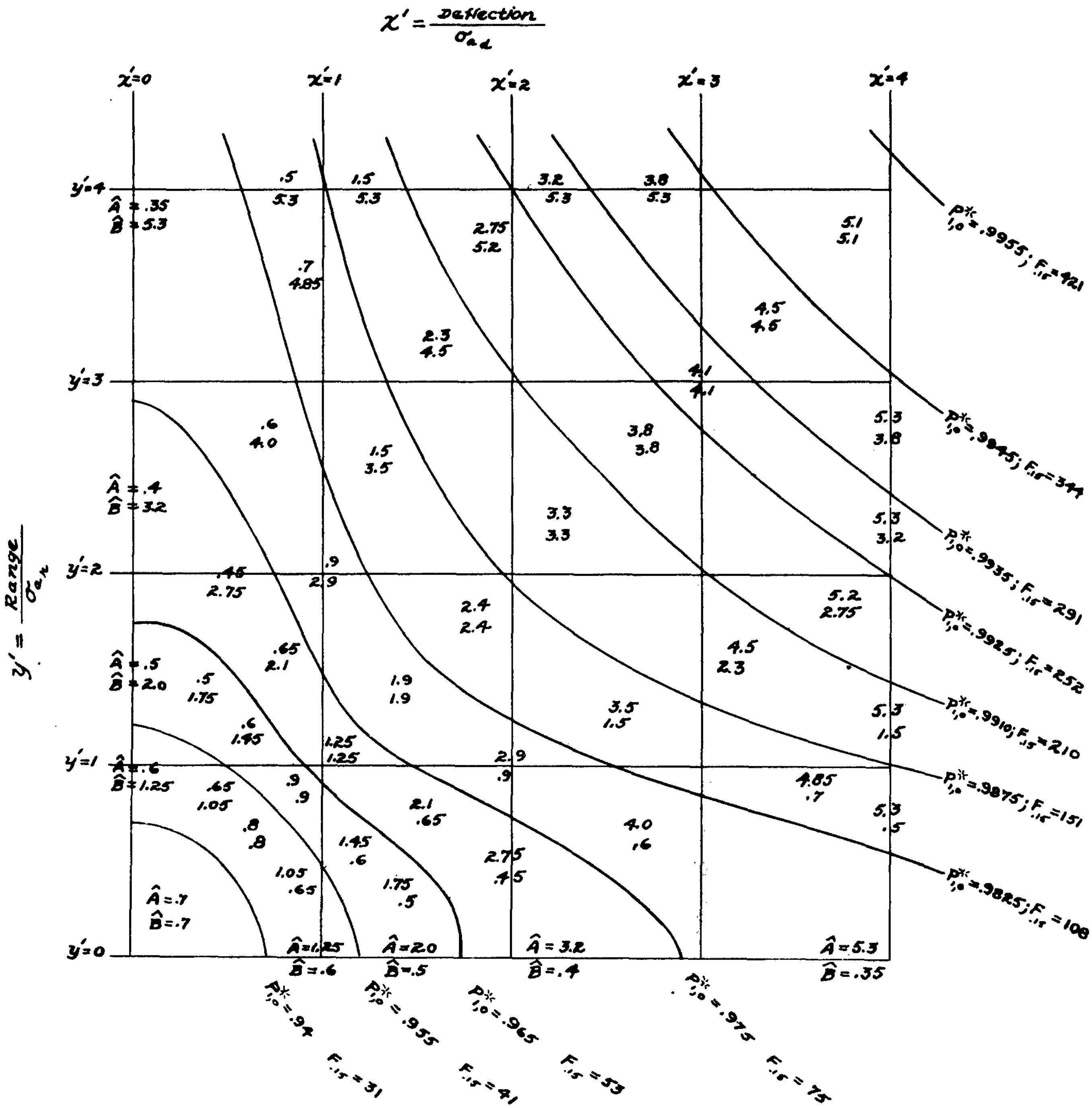


Figure 14

Curves of Optimum $P^{*}_{F,0}$. Values of $\hat{A}$ and $\hat{B}$.

$\Delta = .05$

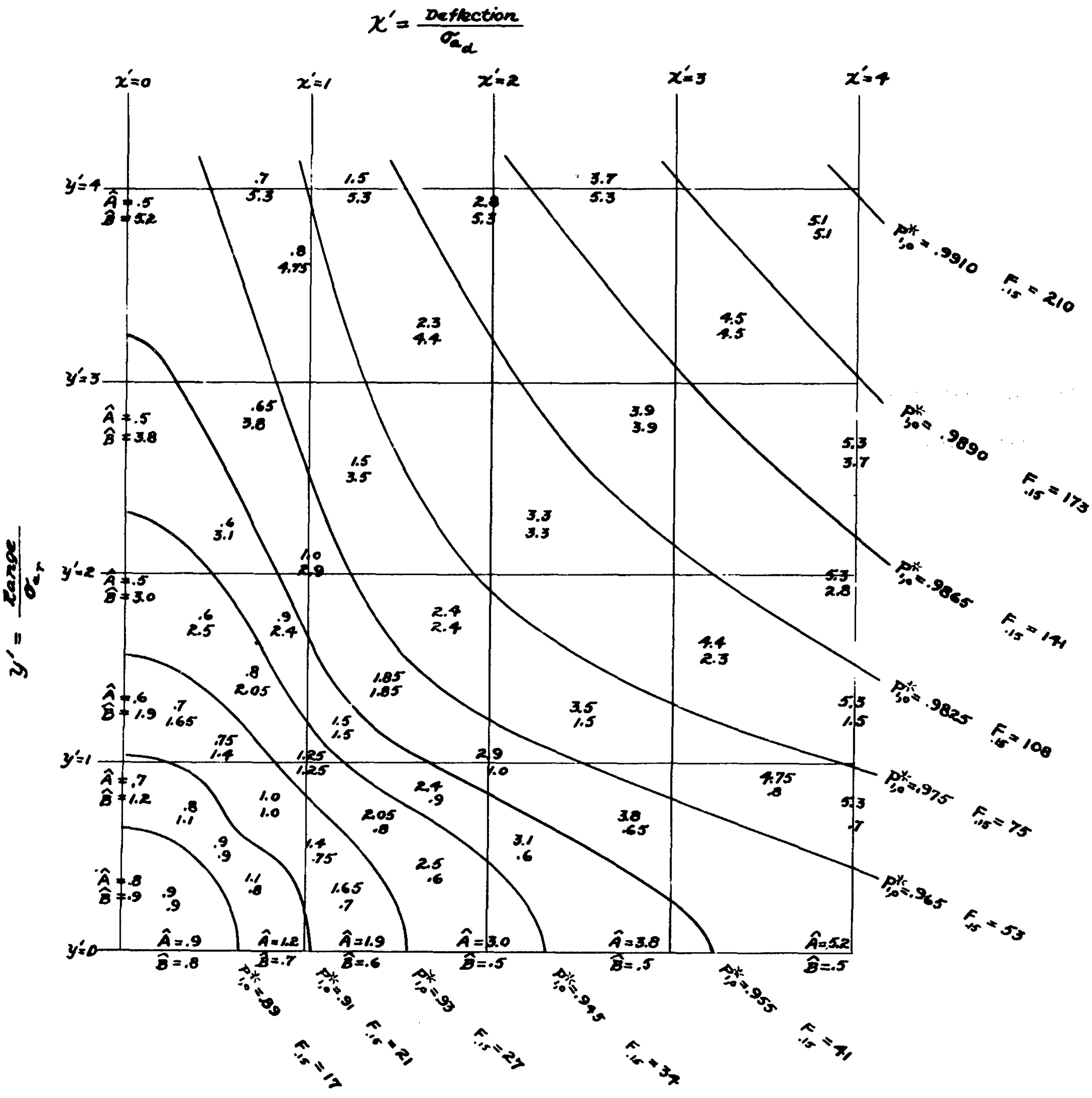


Figure 15

Curves of Optimum $P^*_{F,0}$. Values of $\hat{A}$ and $\hat{B}$.

$\Delta = .10$

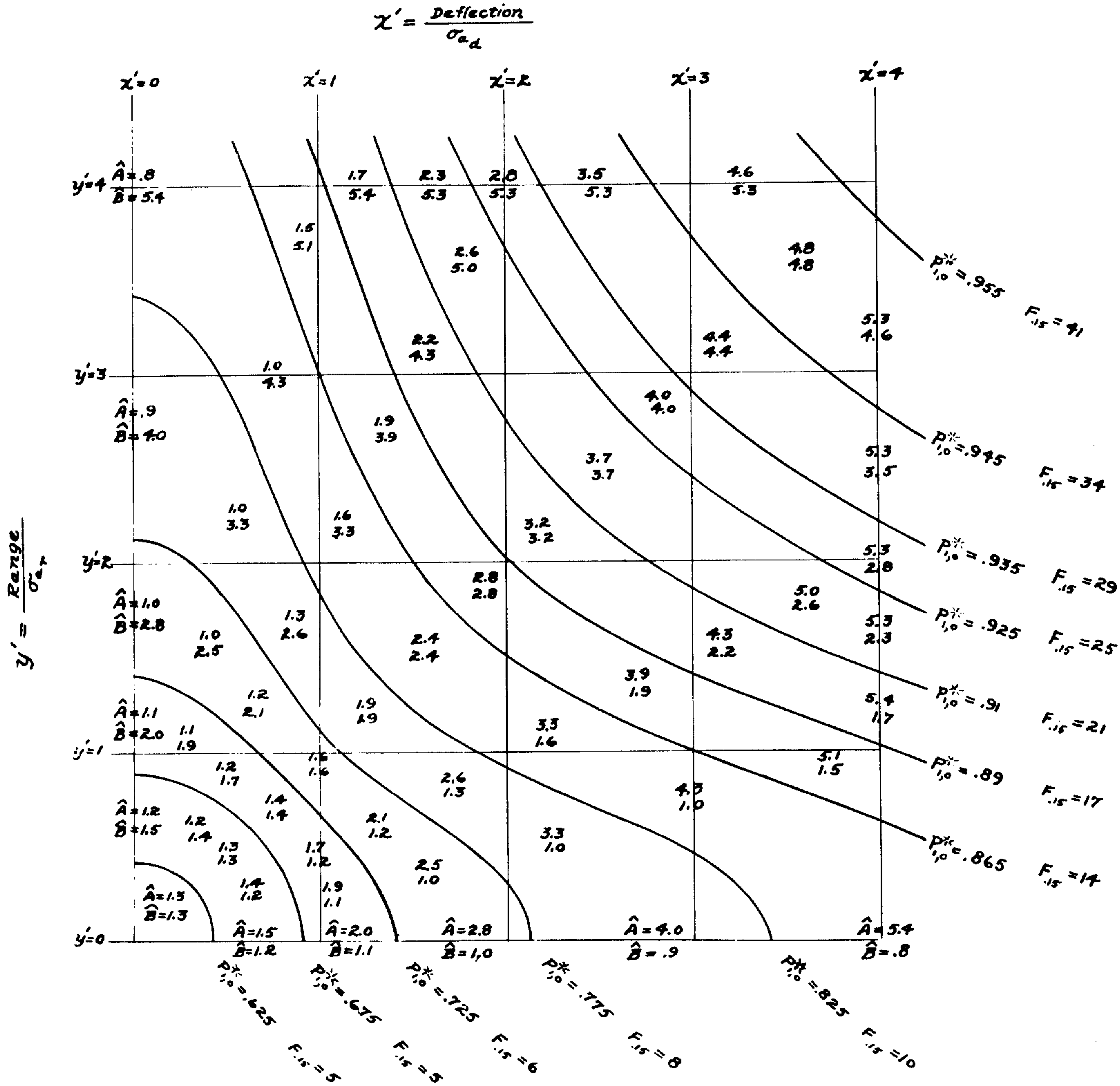


Figure 16

Curves of Optimum $P^{*}_{F,0}$. Values of $\hat{A}$ and $\hat{B}$.

Δ = .50

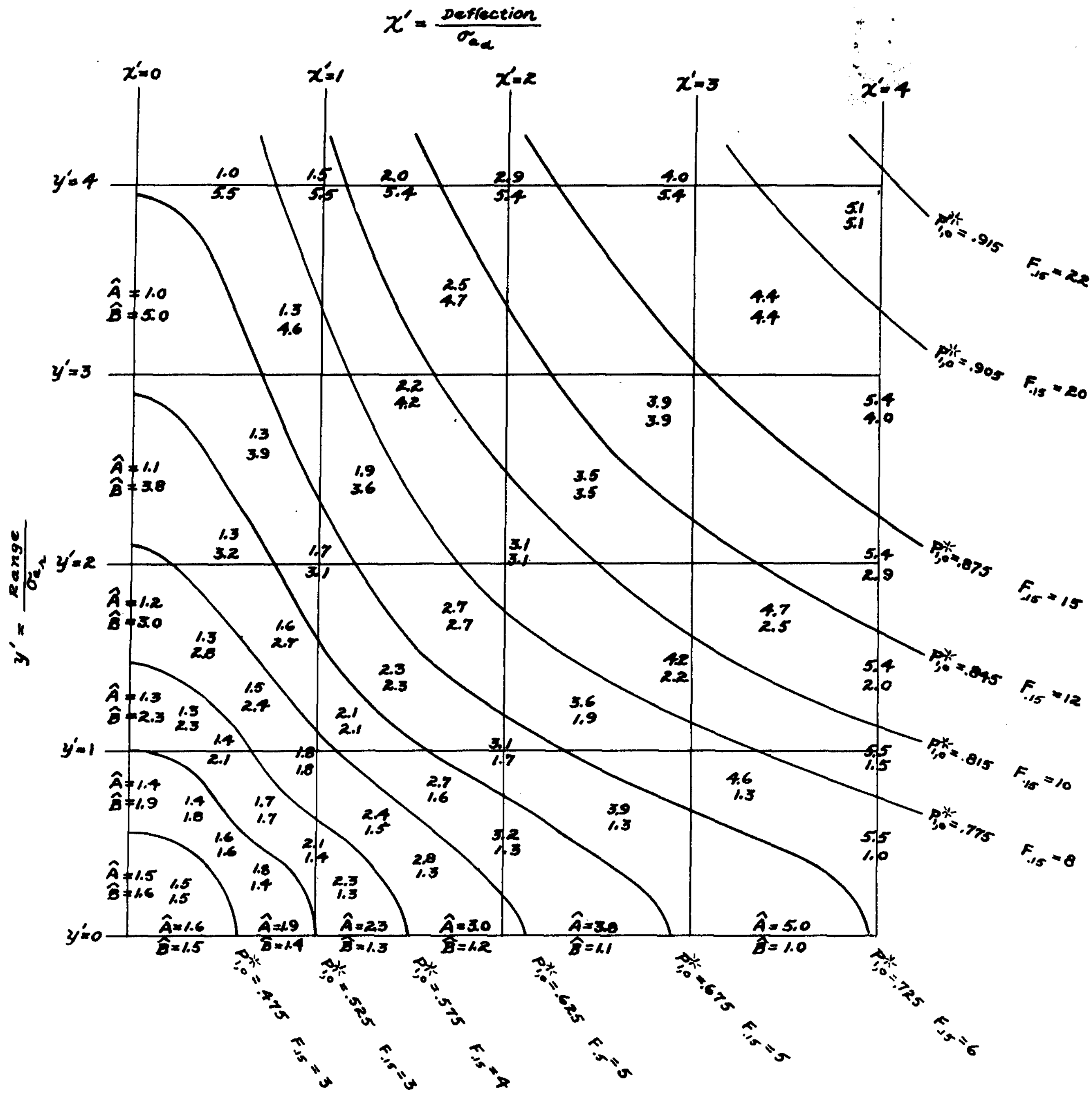


Figure 17

Curves of Optimum $P^{*}_{F,0}$. Values of $\hat{A}$ and $\hat{B}$.

Δ = 1.00

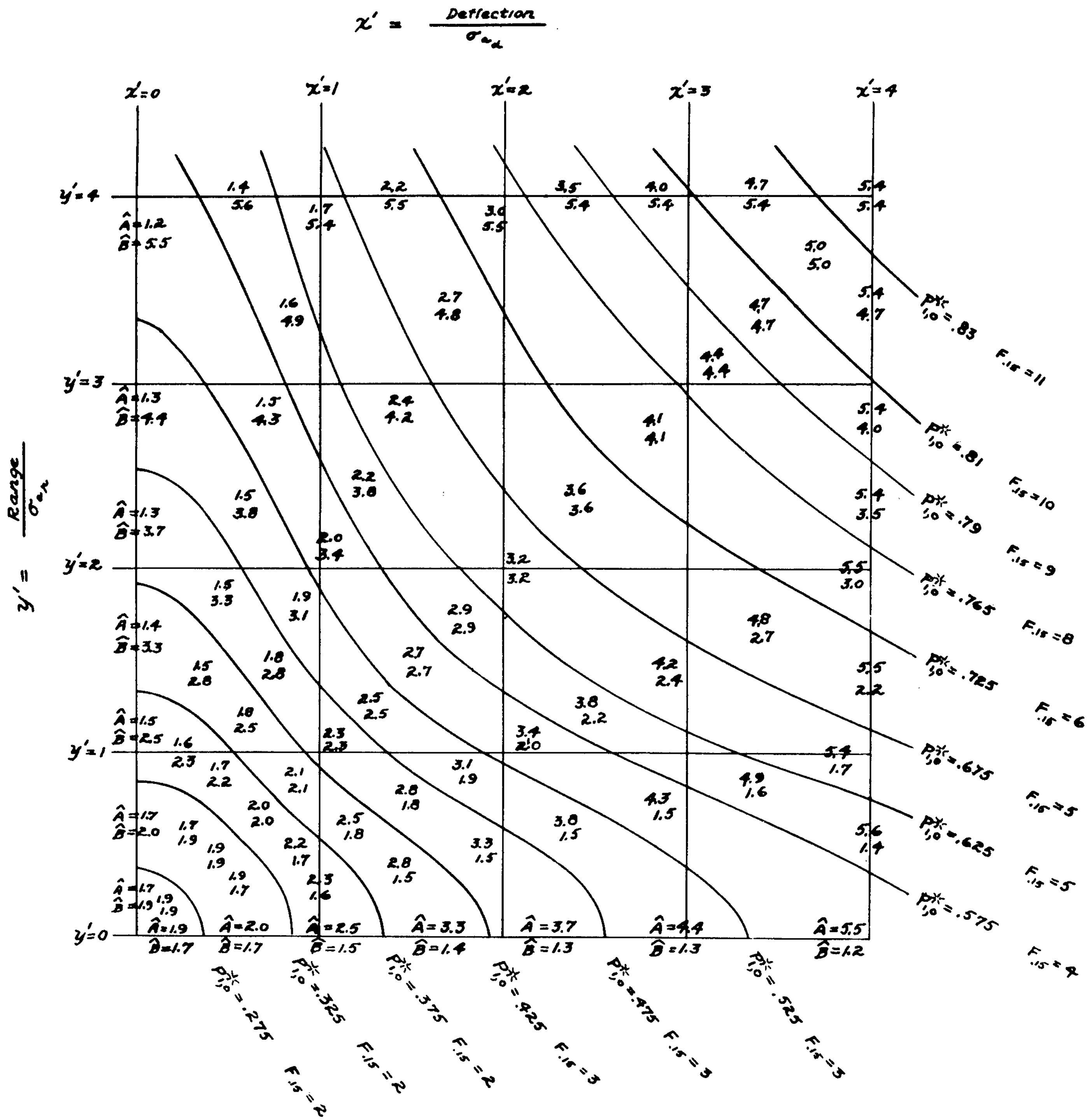


Figure 18

Curves of Optimum $P^*_{F,0}$. Values of $\hat{A}$ and $\hat{F}$

$\Delta = 2.00$

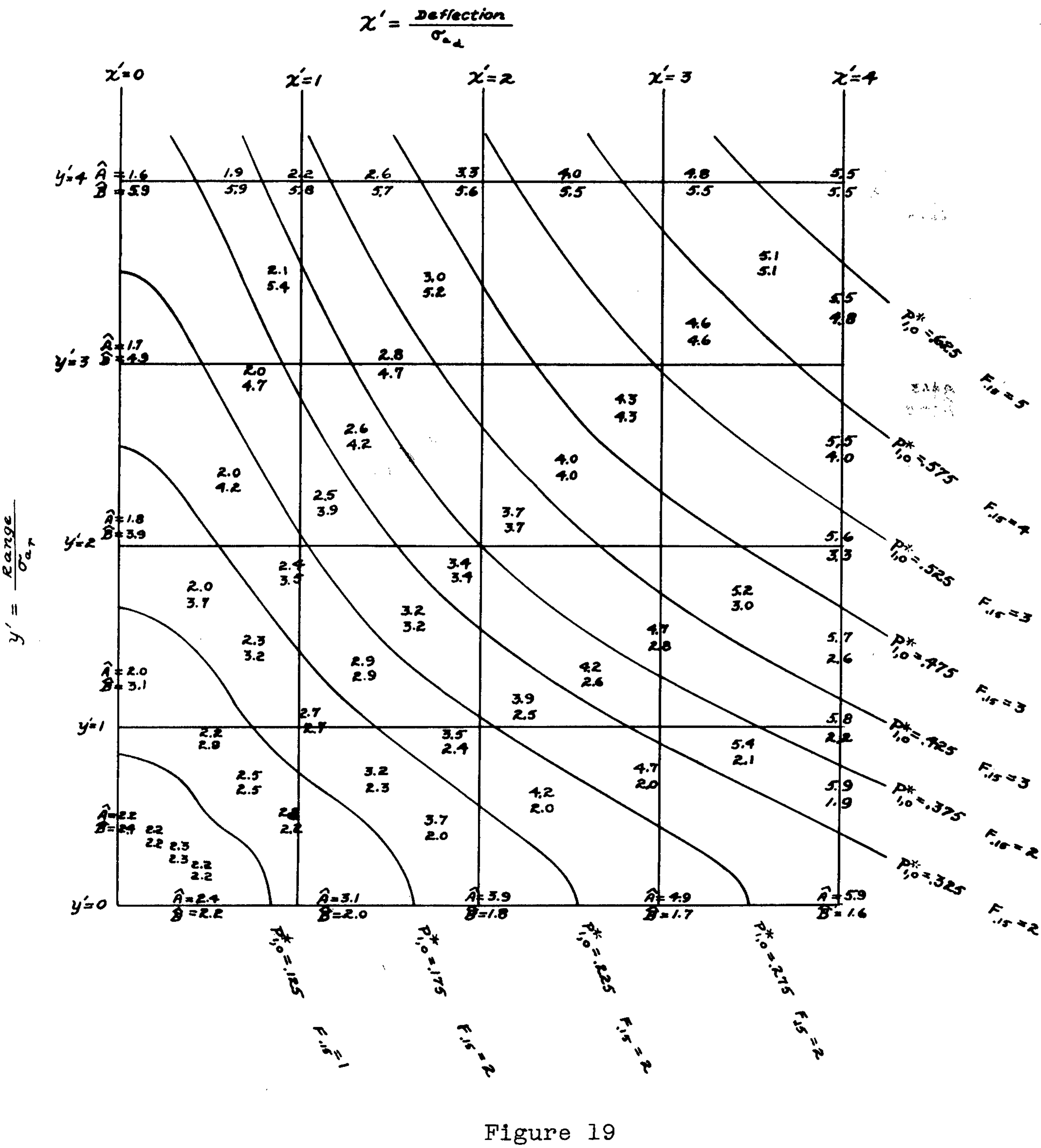


Figure 19

Curves of Optimum $P^*_{F,0}$. Values of $\hat{A}$ and $\hat{B}$.

$\Delta = 5.00$

to all points on the curve and the second the value, say $F_{.15}$, of F such that the minimum $P^{*}_{F,0} \leqq .15$.

Suppose, for example, that, in planning a knockout attack on a rectangular area whose corners have the standardized coordinates $x' = \pm 1.5$ and $y' = \pm 2.0$, the formations and the bombs considered give the value $\Delta = .5$. A glance at Figure 16 shows that, with the bomb pattern minimizing the probability of missing the sub-target in the corner of the target area, this probability will be reduced to something below the level of .15 if at least 14 formations participate in the attack. The approximate standardized half dimensions of the bomb pattern requisite for the purpose may also be interpolated in Figure 16, namely $\hat{A} = 2.2$ and $\hat{B} = 2.9$. The figures are arranged for interpolation along or across a contour zone.

Frequently such values will be found sufficiently accurate with the value of $P^{*}_{F,0}$ reacting very mildly to changes in the pattern dimensions. Unfortunately, the dependence of $\hat{A}$ and $\hat{B}$ on x', y' and Δ has a singularity which is apparent on Chart III. Near this discontinuity the values of $\hat{A}$ and $\hat{B}$ frequently change rapidly and hence are given densely in that region in Figures 13 - 19. These figures are designed to give quick preliminary results, subject to certain recognized inaccuracies in interpolation. Hence it is expedient to confirm these results on Chart III

particularly where the value of Δ for a contemplated mission differs from that given in the figures. It is emphasized that interpolation should not be applied to the values of $F_{.15}$ directly. One should first interpolate the value of $P^{*}_{1,0}$ and then determine $F_{.15}$ by raising the interpolated $P^{*}_{1,0}$ to increasing powers until they fall below .15. The authors realize that the method of determining $F_{.15}$ and $\hat{A}$ and $\hat{B}$ using Figures 13 through 19 presents considerable room for improvement.

Once the optimum values of the standardized half dimensions of the bomb pattern, either $\hat{A}$ and $\hat{B}$, or A^{*} and B^{*} are determined, the optimum half dimensions in feet are obtained from a simple multiplication by the appropriate standard errors of aiming. Since exactly the same method is applicable whether $\hat{A}$, $\hat{B}$ or A^{*}, B^{*} is being considered, it will be summarized only once using $\hat{A}$ and $\hat{B}$. We have

$$A_{opt.} = \hat{A}\cdot\sigma_{a_d}, \qquad B_{opt.} = \hat{B}\cdot\sigma_{a_r} \tag{102}$$

The corresponding optimum spacing of bombs in train is given by the approximate formula

$$2\hat{u} = 2\left\{\frac{(\hat{B}\cdot\sigma^2_{a_r}) - 3(\sigma^2_F + \sigma^2_{d_r})}{n^2 - 1}\right\}^{\frac{1}{2}} \tag{103}$$

and, if the formation is to be composed of s strings of planes,

the optimum lateral spacing between strings is

$$2\hat{d} = 2\left[\frac{(\hat{A}\cdot\sigma_{a_d})^2 - 3(\sigma_f^2 + \sigma_{d_d}^2)}{s^2 - 1}\right]^{\frac{1}{2}}. \tag{104}$$

If the value of $2\hat{d}$ so obtained is large, say greater than $3\{\sigma_f^2 + \sigma_{d_d}^2\}$, and especially if this large value of $2\hat{d}$ goes with a much smaller value of $2\hat{u}$, then this should be considered as an indication that the chosen number s of strings is too small to insure an approximate uniformity of the bomb density within the pattern. In this event, it is recommended to increase s. If in (103) and (104) the numerator of the quantity in brackets is negative, it means that the optimum pattern is obtained by minimizing the corresponding spacing 2u or 2d.

The probability $P^*_{F,0}$ of missing a circular sub-target S(x,y) of radius R with its center at the specified point (x,y) is obtained either from Chart V or Chart Va at the end of this report. Chart V is a little simpler to use than Chart Va, but the range of its arguments is smaller. One of the arguments with which to enter Charts V and Va is obtained from Chart I.

The arguments upon which $P^*_{F,0}$ depends are

$$D_1 = \frac{mR^2}{4AB} = \frac{\Delta}{A'B'} \tag{105}$$

$$I = \{G(x'+A') - G(x'-A')\}\{G(y'+B') - G(y'-B')\} \tag{106}$$

and F = number of formations attacking the target area. While the computation of D_1 can be done with an ordinary slide rule, that of I requires the values of the normal integral G(x). These may be obtained from Chart I.

Instructions for using Charts V and Va are very simple and are given on the charts. It will be seen that if the value of $P^*_{F,0}$ is prescribed, the nomograms will yield the requisite value of F.

The value of Q(x,y) is computed from the formula

$$Q(x,y) = (1-e^{-\pi D_1})\{\Pi(x'+A')-\Pi(x'-A')\}\{\Pi(y'+B')-\Pi(y'-B')\}/x'y' \tag{107}$$

The computation requires the use of Chart II to obtain the values of the function $\Pi(t)$ and of Chart VI giving directly the value of the first factor

$$\Psi(D_1) = 1 - e^{-\pi D_1} \tag{108}$$

It will be remembered that Q(x,y) was given three interpretations. First, Q(x,y) is the probability of one formation hitting a circular sub-target of radius R placed at random within a rectangular target area. The area is centered at the aiming point with corners $\pm x$, $\pm y$. Second, Q(x,y) represents the expected proportion of sub-targets <u>randomly</u> distributed within the same target area as above,

which will be hit in one formation attack. Finally, if the target area is filled with fixed (not random) sub-targets, the density of which is approximately the same over the whole area, then $Q(x,y)$ will be approximately equal to the expected proportion of sub-targets hit in one formation attack.

D. Illustrations.

1. Clearing a path through a mine field by bombing. Eighteen plane formations flying in three strings, with each plane carrying $n = 40$ one hundred pound bombs, are considered for clearing a path 100' wide and 2000' long through a mine field. The bombs are fitted with a special fuze extension increasing their radius of efficiency to $R = 15'$. It is expected that the opposition will be mild and that, therefore, the bombing may be done from a low altitude so that $\sigma_{a_d} = \sigma_{a_r} = 400'$.

It is required to determine (i) the best pattern of the formation and (ii) the number F of formations insuring that the probability of missing the land mine in the corner of the proposed path (which is the most difficult to hit) does not exceed .15.

We begin by computing Δ and the standardized coordinates of the center of the land mine in the corner of the proposed path. We have

$$\Delta = \frac{18 \times 40 \times 15^2}{4 \times 400 \times 400} = .253.$$

Since the aiming point of the center of the bomb pattern is the center of the proposed path, we have

$$x = 50', \quad y = 1000'.$$

Thus

$$x' = \frac{50}{400} = .125, \; y' = \frac{1000}{400} = 2.5.$$

Using Chart III it is found that

$$\hat{A} = .72, \quad \hat{B} = 3.65$$

and therefore

$$A_{opt.} = \hat{A} \cdot \sigma_{a_d} \doteq 288',$$
$$B_{opt.} = \hat{B} \cdot \sigma_{a_r} = 1460'.$$

With the values of the S.E.'s of formation pattern and of bomb dispersion given in subsection VII-A, and assuming tentatively that the formations will be composed of $s = 3$ strings of planes, we find

$$2\hat{d} \doteq 2\left\{\frac{288^2 - 3(100^2 + 40^2)}{3^2 - 1}\right\}^{\frac{1}{2}} = 155'$$

which represents a lateral spacing of planes not infrequent in actual practice and not too large from the point of view of computation of the bomb pattern. Finally, the optimum spacing of

bombs in train

$$\hat{2u} = 2\left\{\frac{1460^2 - 3(500^2 + 40^2)}{40^2 - 1}\right\}^{\frac{1}{2}} = 59' = 60' \text{ (say)}.$$

In order to find the number F of formations assuring that the probability of missing the corner mine does not exceed the limit of .15, Chart V or Chart Va may be used. The arguments with which to enter these charts are D_1 and I.

$$D_1 = \frac{N \times n \times R^2}{4 \cdot A_{opt.} B_{opt.}} = \frac{\Delta}{\hat{A}\hat{B}} = .0963.$$

The argument I is computed using Chart I. We have

$$x'+\hat{A} = .845, \quad G(.845) = .301$$
$$x'-\hat{A} = -.595, \quad G(-.595) = -.224$$

Difference .525

$$y'+\hat{B} = 6.15, \quad G(6.15) = .500$$
$$y'-\hat{B} = -1.15, \quad G(-1.15) = -.375$$

Difference .875

Thus $$I = .525 \times .875 = .459$$

Entering Chart V or Chart Va with the above values of $D_1 = .0963$ and $I = .459$ it is found that, to reduce the probability $P^*_{F,0}$ of missing a land mine at the corner of the proposed path to the desired level $P^*_{F,0} \leq .15$, a total of $F = 15$ formations, of

18 planes each, will be needed.

For the sake of illustration we will now perform the computations from the point of view of minimizing the chance of missing a mine placed at the center of the proposed path. The half dimensions of the pattern which optimize the probability of hitting $x' = y' = 0$, denoted by A_o and B_o, are

$$A_o = B_o = 424'.$$

This result leads to the optimum lateral spacing of strings of planes

$$2\hat{d}_o = 269' = 270' \text{ (say)}$$

which is considerably larger than in the previous case, but not excessively large. However, the attempt to compute the optimum spacing of bombs is unsuccessful. In fact, we have

$$2\hat{u}_\sigma = 2\left\{\frac{179,776 - 754,800}{40^2 - 1}\right\}^{\frac{1}{2}}$$

and it is seen that the figure in the numerator under the square root is negative. It follows that, with the assumed S.E.'s of pattern and bomb dispersion, it is impossible to achieve a bomb pattern of such short length as $2B_o = 848'$. The best that can be done is for all planes to release their bombs in salvo, putting $2u = 0$. Then the half dimension B of the bomb pattern will be

$$B = \{0 + 3(\sigma_F^2 + \sigma_{d_r}^2)\}^{\frac{1}{2}} = 869'.$$

In the problem of clearing a path through a mine field it is doubtful that a bomb pattern will be desired minimizing the chance of missing the mine at the center of the pattern, neglecting all other mines. The above computations are given solely for illustrative purposes. For the same purpose, the little table below was computed. It gives the probabilities of missing a mine at a few points along the border of the proposed path, corresponding to two bomb patterns:

Bomb Pattern	Dimensions	2d	2u
I (Optimum for corner)	576' x 2920'	155'	60'
II(Compromise optimum for Center)	848' x 1738'	270'	0

The first pattern is optimum for the corner mine and the second has dimensions as close as possible to those optimum for the center of the proposed path.

Pattern x	Pattern y	I $P^*_{15,0}$	II $P^*_{15,0}$
50'	0	.109	.038
50'	200'	.109	.040
50'	400'	.110	.049
50'	600'	.118	.083
50'	800'	.126	.150
50'	1000'	.149	.302

It will be seen that while the second pattern is much more efficient than the first for land mines within 600' from the center of the proposed path, beyond that distance the situation is reversed. Thus, if it is desired to create a path 2000' long which is reasonably clear of land mines, the first formation pattern will be found more advantageous than the second.

In order to obtain a more distinct idea of possible results of an attempt to clear a path through a mine field by bombing, when the probability of missing a mine at the corner of the proposed path is fixed at .15, a sampling experiment was carried out. Figure 20 gives the bomb plot obtained. Here the intended path has the dimensions 100' x 800'. It is marked by a rectangle in the middle of the plot. The original of this diagram was considerably larger (1" = 40') so that what in the present form looks like a dot was a circle carefully drawn to scale, with its radius equal to the assumed radius of efficiency of the bombs and with its center at the point of impact of the bomb. Thus the area covered by partly overlapping dots should be considered as cleared of mines.

It will be seen that the fixed value of the probability .15 for the corner mine corresponds to a good clearance not only of the rectangle representing the path, but also of a considerable area surrounding it.

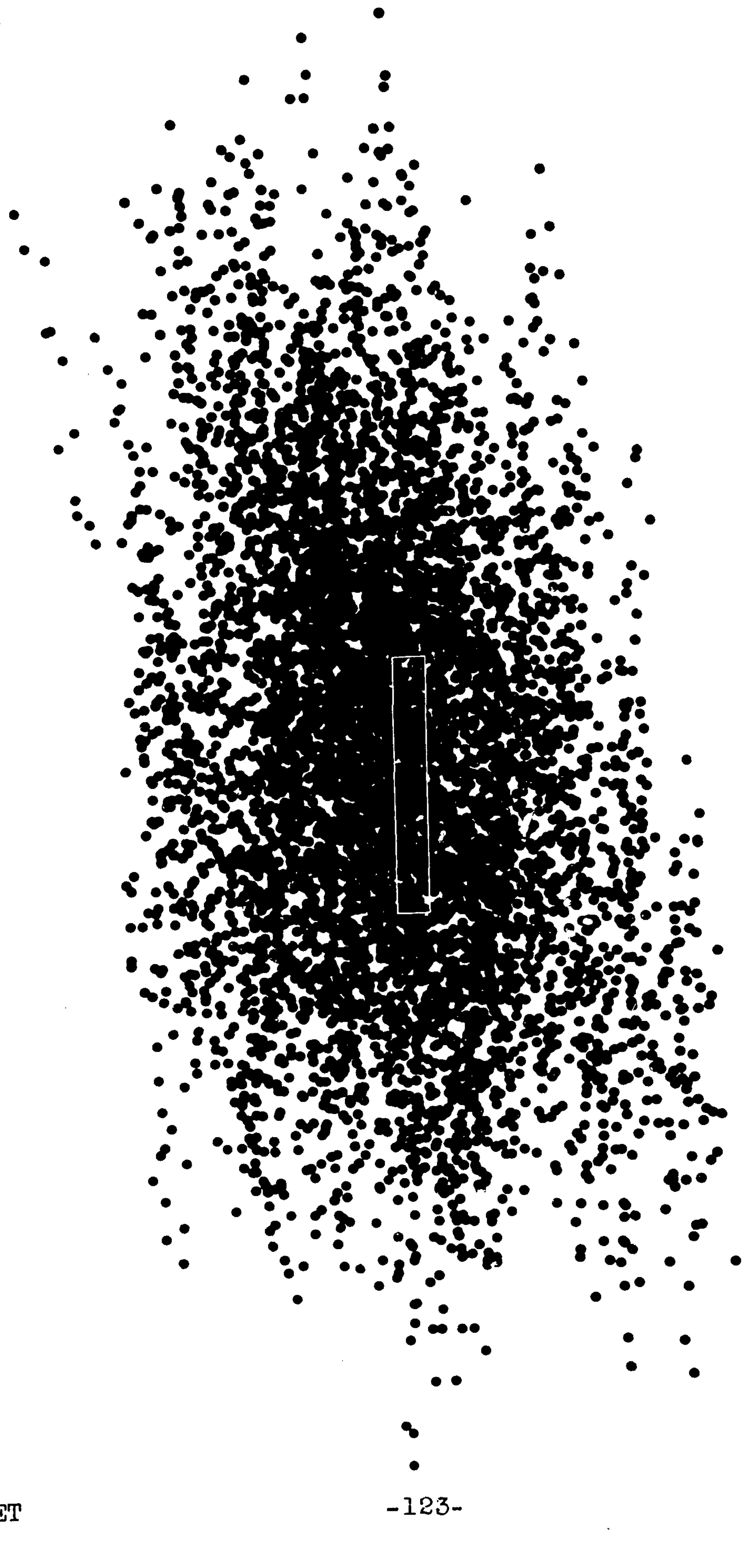


Figure 20

Bomb Plot Corresponding to $P^*_{F,0} = .15$

The experiment was carried out assuming $\sigma_{a_d} = \sigma_{a_r} = 400'$, $\sigma_f = 100'$, $\sigma_F = 500'$, $\sigma_{d_d} = 30'$, $\sigma_{d_r} = 50'$. Also, in order to obtain a more realistic picture of what may happen in practice, it was assumed that the direction of the line of flight of single planes as well as of whole formations is subject to variation with S.E.'s equal to 1° and 8° respectively. The values of other parameters are $n = 12$, $R = 16$, $F = 29$, $N = 18$, and $2u = 100'$.

In discussing the preceding example the emphasis is laid on arithmetical procedures leading to the optimum pattern. In the following example no details of arithmetic are given, only the final results.

2. An attack on a dispersal area.

Figure 21

Ballale Island

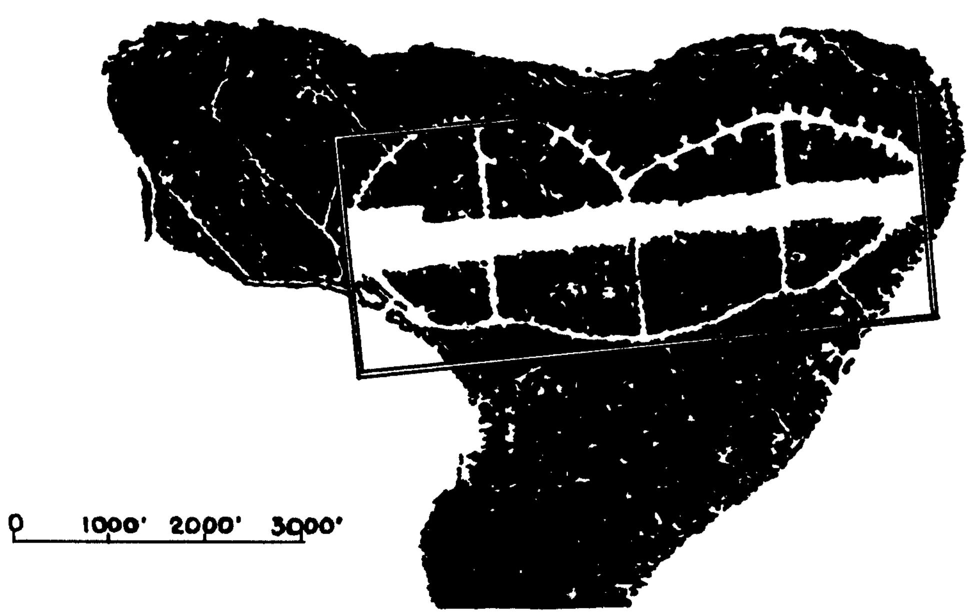


Figure 21 represents a map of Ballale Island sketched from IMPACT, Vol. 1, No. 9, 1943, where it is given as a typical Japanese air base in the Pacific. We will consider the problem of selecting the best method of attacking enemy aircraft that may be parked in the revetments and in the dispersal area surrounding the runway. We will consider formations composed of $N = 12$ planes and assume that each plane releases $n = 144$ fragmentation bombs so that the total number of bombs released by one formation is $m = 1728$. The radius of efficiency of bombs will be taken as $R = 60'$.

The problem of the best pattern will be considered in both of its aspects, relating to the preparation for landing operations when it is desired to deliver a knockout blow and in reference to routine bombing where it is desired to insure the most economical use of the bombers available. The standard errors of aiming will be assumed to be

$$\sigma_{a_d} = 386', \ \sigma_{a_r} = 1519'$$

consistent with the last line of the table on p. 94, relating to the experience of the Thirteenth Air Force.

Taking the center of the runway as the aiming point and assuming that the direction of flight is parallel to the axis of the runway a rectangle is drawn with one of its sides parallel to the line of flight covering the whole area which may be presumed to include all the parking places of the aircraft and in

general all of the desirable sub-targets. The corners of the rectangle actually drawn (the appropriateness of this is not insisted on in any way) have the coordinates $x = \pm 1250'$ and $y = \pm 3050'$, with their standardized values approximately

$$x' = 3.25 \quad \text{and} \quad y' = 2.0.$$

Following the procedure explained in detail in the preceding example, it is found that for this particular target area the optimum $\hat{A}$ and $\hat{B}$ corresponding to its corner are

$$\hat{A}_c = 4.80, \quad \hat{B}_c = 3.14,$$

those corresponding to the center of target

$$\hat{A}_o = \hat{B}_o = 1.91,$$

and finally

$$A^* = 3.34, \; B^* = 2.22.$$

The smallest number F of formation attacks which reduce the probability of missing a sub-target in the corner of the target area to a level not exceeding .15, with the half dimensions of the pattern $\hat{A}_c$, $\hat{B}_c$, is $F = 5$.

Table XVI gives the values of $P^*_{5,0}$ for a network of points covering one quadrant of the target area, computed using the three systems of pattern dimensions just found. It will be seen that with the half dimensions of the bomb pattern $\hat{A}_o$, $\hat{B}_o$ the sub-targets which may be located near the corners of the target area will be relatively safe from hits. On the other hand

TABLE XVI

Values of $P^*_{5,0}$ Computed Using Three Alternative Bomb Patterns.

PART I. $\hat{A}_c = 4.80$, $\hat{B}_c = 3.14$. Optimum for Knockout Attack, Corner of Target Area

y' \ x'	.0	.5	1.0	1.5	2.0	2.5	3.0	Proportion of Sub-Targets Missed
0	.063	.063	.063	.064	.064	.066	.072	
.5	.064	.064	.064	.064	.065	.067	.073	
1.0	.067	.067	.067	.067	.068	.070	.076	.080
1.5	.076	.076	.076	.076	.076	.078	.085	
2.0	.099	.099	.099	.099	.099	.102	.110	

PART II. $\hat{A}_o = \hat{B}_o = 1.91$. Optimum for Knockout Attack, Center of Target Area

y' \ x'	.0	.5	1.0	1.5	2.0	2.5	3.0	Proportion of Sub-Targets Missed
0	.000	.001	.003	.017	.082	.261	.537	
.5	.001	.001	.004	.021	.091	.275	.549	
1.0	.003	.004	.010	.037	.125	.320	.587	.190
1.5	.017	.021	.037	.085	.201	.407	.654	
2.0	.082	.091	.125	.201	.342	.541	.744	

PART III. $A^* = 3.34$, $B^* = 2.22$. Optimum for Single Routine Attack of Target Area

y' \ x'	.0	.5	1.0	1.5	2.0	2.5	3.0	Proportion of Sub-Targets Missed
0	.005	.005	.005	.006	.011	.024	.068	
.5	.006	.006	.006	.008	.012	.027	.073	
1.0	.010	.011	.011	.013	.019	.038	.092	.043
1.5	.027	.027	.028	.032	.042	.070	.139	
2.0	.080	.081	.083	.089	.107	.149	.236	

the neighborhood of the center of the runway is likely to be over-bombed. The half dimensions A^* and B^* appear to be much more satisfactory. But the real coverage of the whole area recognized as the target is attained when the optimum half dimensions for the corner $\hat{A}_c = 4.80$ and $\hat{B}_c = 3.14$ are used.

Apart from the individual probabilities referring to particular points where a sub-target may be located, Table XVI also gives the average values, representing approximately the expected proportion of targets which will be missed in $F = 5$ formation attacks. The proportion given for each pattern represents the approximate value of an integral over the entire target area. It is a weighted average of all figures in the quadrant with weights equal to:

(i) Unity for $(x' = y' = 0)$ and $(x' = 0,\ y' = 2)$,

(ii) 2 for all other entries with $x' = 0$, $y' = 0$, or $y' = 2$,

(iii) 4 for all other entries.

As would be expected, the best pattern appears to be that given by A^* and B^*. However, the average corresponding to the half dimensions $\hat{A}_c$ and $\hat{B}_c$ is not much worse. In this connection it may be asked whether or not the two problems of optimum pattern, one having in view a knockout assault and the other routine bombing, are significantly different. In fact, it may be suspected that (a) as a rule the values $\hat{A}_c$ and $\hat{B}_c$ will yield an expected proportion

of targets destroyed that is nearly equal to the proportion given by A^* and B^*; and (b) that the missions planned to deliver a knockout blow are about as economical as those suggested as optimum for routine bombing. A negative answer to question (a) is obtained on inspecting Table XVIII in the following illustration 3. A negative answer to question (b) is obtained by comparing the expected consequences of using the same force of planes for a knockout attack and for several routine missions.

Table XVII

Expected Proportion of Sub-Targets Hit in a Single Formation Attack

Half Dimensions of the Bomb Pattern	$Q(3.25 \cdot \sigma_{a_d}, 2\sigma_{a_r})$
$\hat{A}_c = 4.80$, $\hat{B}_c = 3.14$	.408
$\hat{A}_o = 1.91$, $\hat{B}_o = 1.91$	.401
$A^* = 3.34$, $B^* = 2.22$	.511

It is seen that, if <u>single</u> formations attack the contemplated target area then, with pattern half dimensions A^* and B^*, they may be expected to destroy about 25 percent more sub-targets than if the pat tern half dimensions are $\hat{A}_c$ and $\hat{B}_c$. Also it is seen that if five formations using A^* and B^* are sent to attack <u>five different</u> air bases (routine bombing) each air base harboring

the same number M of sub-targets, then the expected number of sub-targets hit on all five targets will be $5 \times M \times .511 = M \times 2.555$. On the other hand, if the same five formations concentrate their attacks on one air base then this base is likely to be totally destroyed, but the total number of sub-targets hit cannot exceed M and may be expected to amount to $M \times .95$.

These results suggest the conclusion that, from the point of view of the total number of sub-targets hit, concentrated knockout attacks may be much less economical than routine attacks of single formations against single target areas. Also, at least in some cases, the use of pattern dimensions based on A^* and B^* may result in a considerable gain in the number of sub-targets destroyed.

Let $2\hat{d}$ and $2\hat{u}$, computed from $\hat{A}_c$ and $\hat{B}_c$, denote the lateral spacing of planes and the spacing of bombs in train which are optimum for the knockout attack. Similarly, $2d^*$ and $2u^*$, computed from A^* and B^*, will denote the optima for routine bombing. As both $\hat{A}$ and A^* are large, to achieve a more or less uniform density of bombs all over the intended patterns, it will be necessary for all the $N = 12$ planes to fly abreast* or in a single "Vee".

*The authors are not sure whether a formation pattern of this kind is consistent with considerations of safety from enemy opposition. However, considerations of safety are beyond the scope of this report. Also, it seems that in certain theaters the opposition is weak.

Thus s = 12. Since the 144 fragmentation bombs are released in 24 clusters, 6 bombs per cluster, we put n = 24. With these values of s and n it is found

$$2\hat{d} = 307' \text{ (say } 300'\text{)}; \quad 2\hat{u} = 391' \text{ (say } 390'\text{)};$$
$$2d^{*} = 212' \text{ (say } 210'\text{)}; \quad 2u^{*} = 271' \text{ (say } 270'\text{)}.$$

It is emphasized that the above optima depend on the size of the target area, on the direction of flight, on the standard errors of aiming and on the radius of efficiency assumed. The reader may find it interesting to repeat the computations assuming that the direction of flight is not along but across the long axis of the target area.

3. Effect of the size of a formation on its efficiency.

Reports from the various theaters indicate that the number of planes participating in a formation releasing their bombs on the leader varies within very broad limits, from N = 6 (or, perhaps, even N = 3) to N = 36. No doubt, the choice of the size of a formation is made on various grounds which are beyond the scope of this report. However, it seems interesting to inquire whether a mere change in the number N of planes per formation may have an effect on the results of bombing. In order to obtain a set of figures relevant to this question consider the following set of general conditions.

Precision of aiming: $\sigma_{a_d} = \sigma_{a_r} = 1000'$

Bomb: 500 lb. G. P. with assumed radius of efficiency $R = 100'$

Number of Bombs per Plane $n = 12$

Target Area: square $3000' \times 3000'$

Results referring to the above conditions, which are likely to be similar to those prevailing in some missions in the European theater, are given in Table XVIII.

The third column in Table XVIII gives the common value in feet of the two equal dimensions of the bomb pattern, computed from $\hat{A}_c = \hat{B}_c$, from $\hat{A}_o = \hat{B}_o$ and from $A^* = B^*$. Of course, due to the fact that the target area is a square and that $\sigma_{a_d} = \sigma_{a_r}$, the optimum bomb pattern is also a square. Referring to question (a) in illustration 2, it will be noticed that in the present case the pattern dimensions optimizing the routine bombing attacks, i.e. those based on $A^* = B^*$ are closer to those based on $\hat{A}_o = \hat{B}_o$ than to those based on $\hat{A}_c = \hat{B}_c$, which shows that the patterns best for a knockout attack are, in general, essentially different from those most suitable for routine bombing.

The fourth and the fifth columns of Table XVIII refer to the question of efficiency of formations of different size in knockout attacks. The column of $F_{.15}$ gives the number of independent formation attacks needed to reduce the probability of any one sub-target being missed to the arbitrary low level of .15.

TABLE XVIII

Optimum Dimensions of Bomb Pattern and Other Characteristics in Relation to the Number N of Planes in a Formation

N	Optimum Pattern Dimensions		$F_{.15}$	$NF_{.15}$	Q	100Q/N
	Based on	In feet				
6	$\hat{A}_c = \hat{B}_c$	4188	29	174	.095	1.59
	$\hat{A}_o = \hat{B}_o$	1934			.124	2.06
	$A^* = B^*$	2314			.127	2.11
9	$\hat{A}_c = \hat{B}_c$	4366	20	180	.133	1.48
	$\hat{A}_o = \hat{B}_o$	2144			.170	1.88
	$A^* = B^*$	2552			.174	1.93
12	$\hat{A}_c = \hat{B}_c$	4446	15	180	.169	1.41
	$\hat{A}_o = \hat{B}_o$	2314			.210	1.75
	$A^* = A^*$	2732			.216	1.80
36	$\hat{A}_c = \hat{B}_c$	4946	6	216	.378	1.05
	$\hat{A}_o = \hat{B}_o$	3050			.425	1.18
	$A^* = B^*$	3570			.440	1.22

As would be expected, as far as the number of attacks is concerned, an important criterion due to the time element involved, the larger formations are much more effective than the smaller ones. However, apart from the question of time needed for a knockout attack the question may arise of the total number of sorties, $N \cdot F_{15}$. These are given in the fifth column of the table and it will be seen that to reduce the probability $P^{*}_{F,0}$ to the level .15 or below the largest formations of N = 36 planes will require a total of 216 sorties as against 174 sorties required by formations of N = 6 planes, or about 24 percent more.

The last two columns treat a similar question relating to single formation attacks in routine bombing. Each figure in the last column represents the expected proportion of sub-targets within the target areas which will be hit in a single formation attack, averaged per 100 planes participating in missions. It will be seen that from this point of view the smaller formations have an advantage over the larger ones which is even more distinct than above. In fact it appears that the efficiency of the six plane formations, flying their optimum pattern, exceeds that of large formations of N = 36 planes by about 73 percent.

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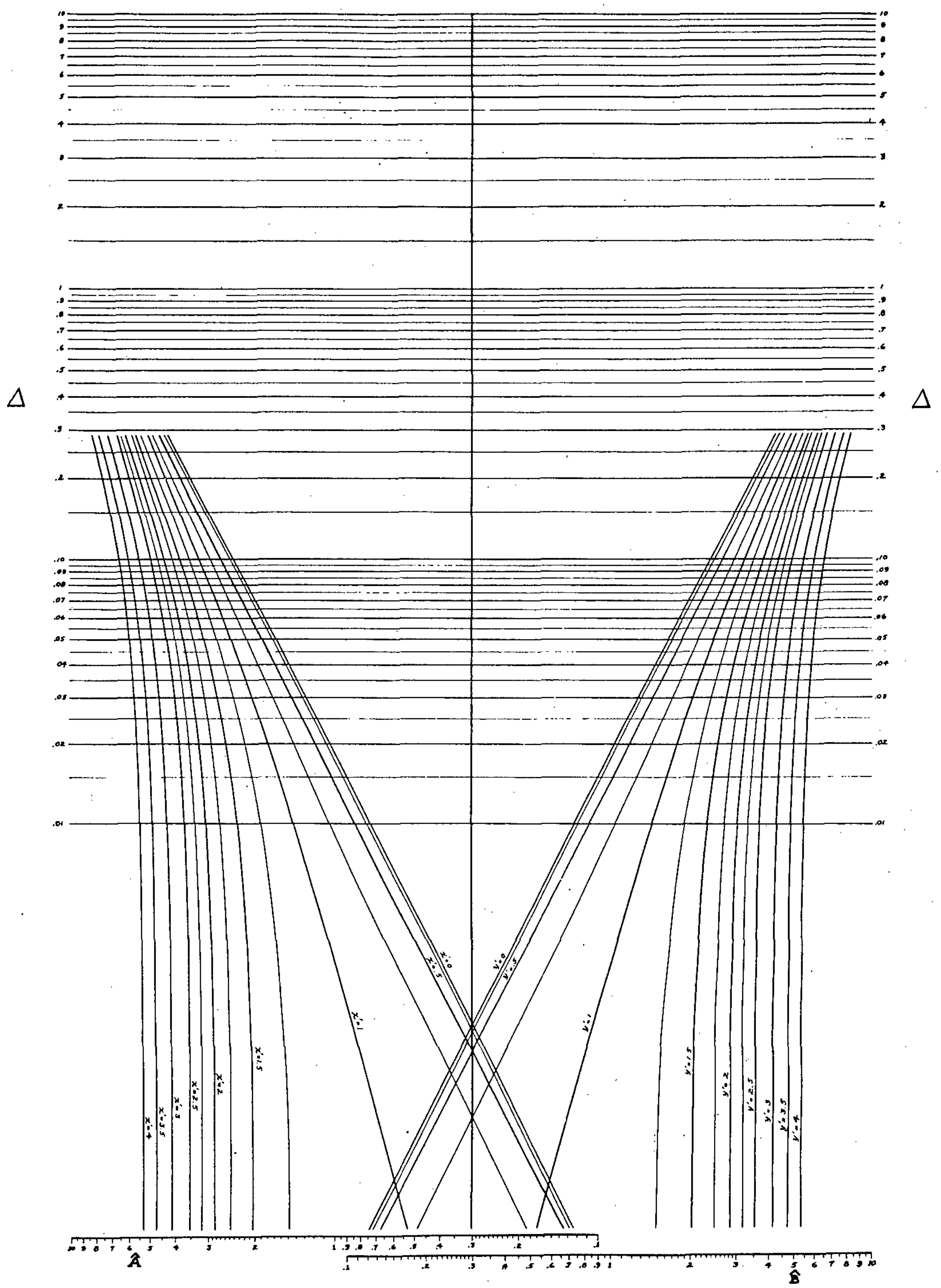
CHART III

NOMOGRAM FOR COMPUTING HALF DIMENSIONS $\hat{A}$ AND $\hat{B}$ FOR x, y, AND Δ



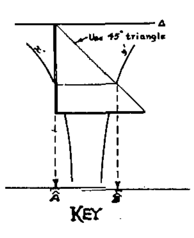

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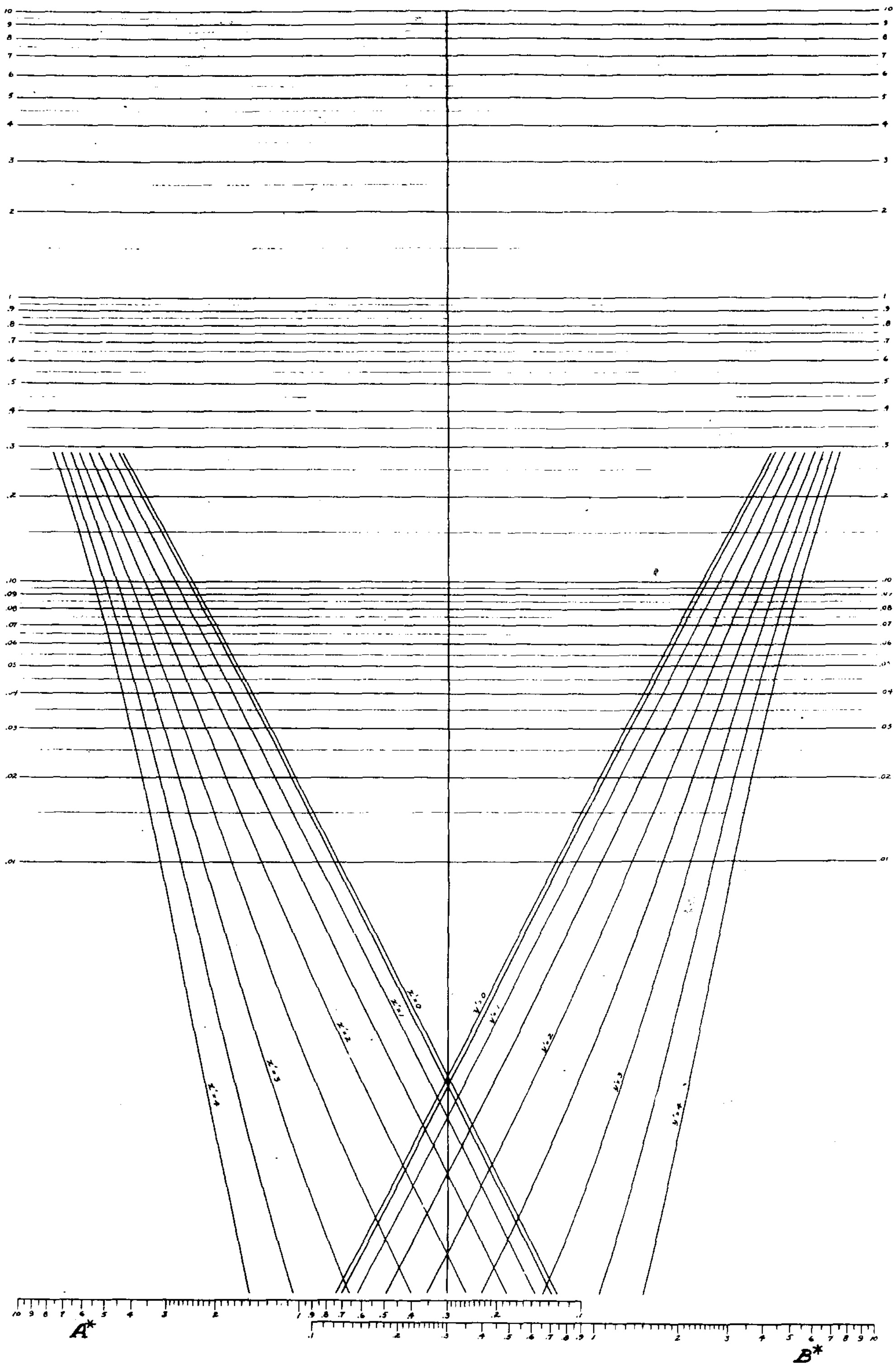
CHART IV [redacted]

NOMOGRAM FOR COMPUTING HALF DIMENSIONS A^* AND B^*



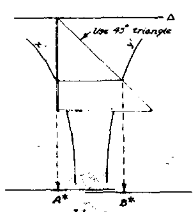


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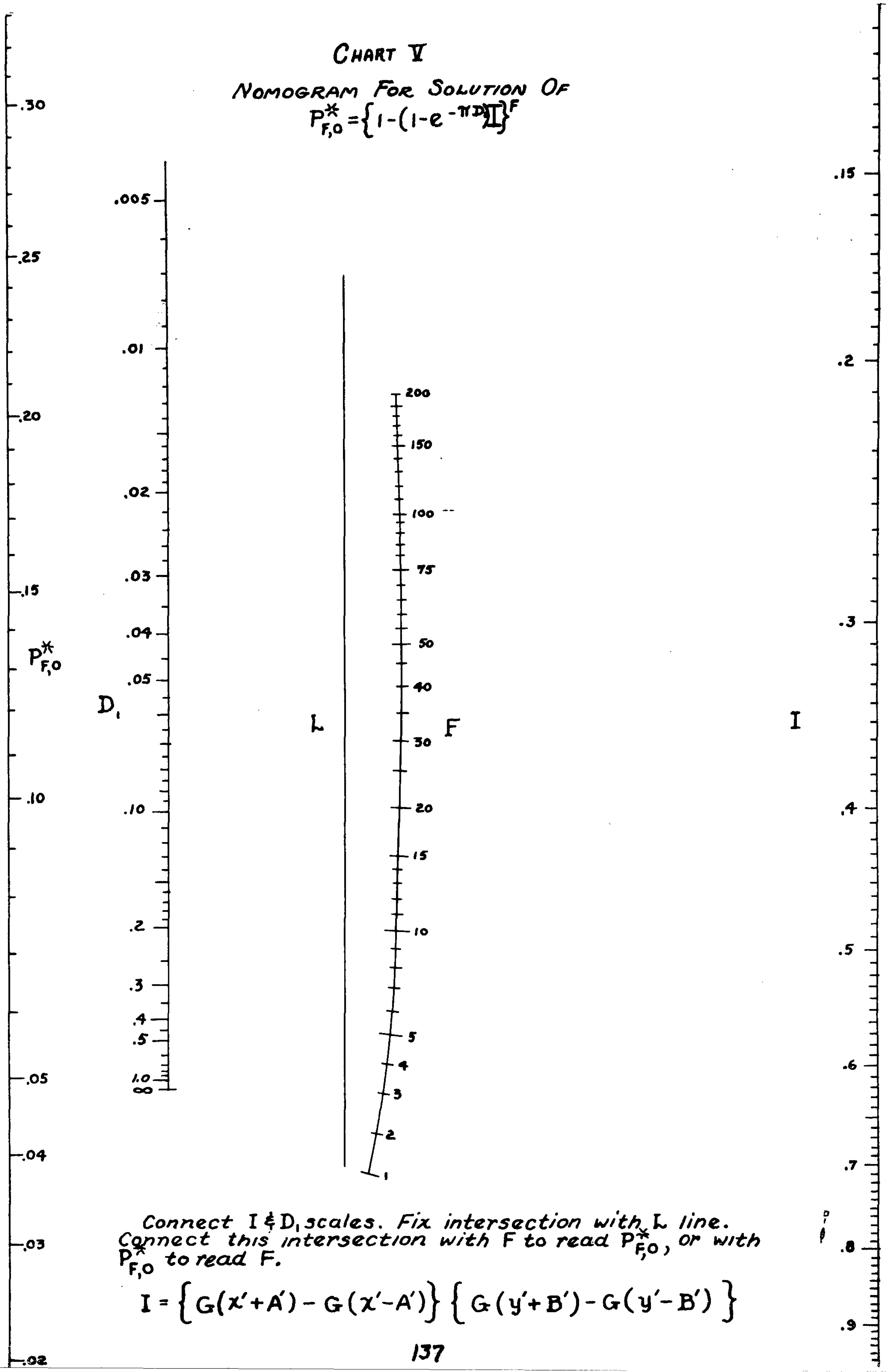
CHART V

NOMOGRAM FOR SOLUTION OF

$$P^*_{F,0} = \left\{1-\left(1-e^{-\pi D_1}\right)I\right\}^F$$



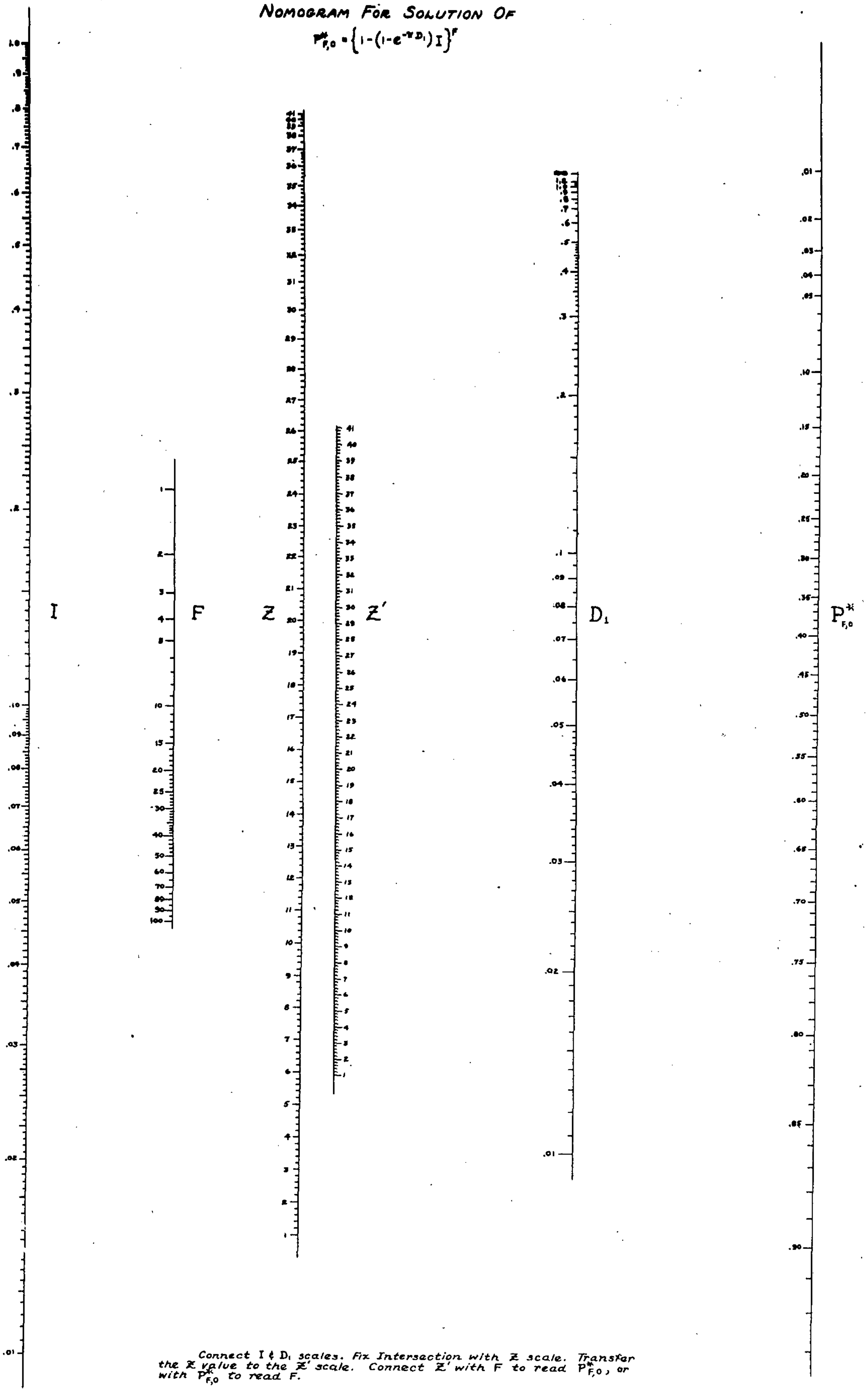
Connect I & D_1 scales. Fix intersection with L line. Connect this intersection with F to read $P^*_{F,0}$, or with $P^*_{F,0}$ to read F.

$$I = \left\{G(x'+A') - G(x'-A')\right\}\left\{G(y'+B') - G(y'-B')\right\}$$

Chart V6

Nomogram For Solution Of

$$P^{*}_{F,0} = \left\{1 - \left(1 - e^{-\nu D_1}\right) I\right\}^{F}$$



Connect I & D_1 scales. Fix Intersection with Z scale. Transfer the Z value to the Z′ scale. Connect Z′ with F to read $P^{*}_{F,0}$, or with $P^{*}_{F,0}$ to read F.

$$I = \left\{G(x'+A') - G(x'-A')\right\}\left\{G(y'+B') - G(y'-B')\right\}$$

CHART VI